

FINITE ORDER SPREADING MODELS

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ABSTRACT. Extending the classical notion of the spreading model, the k -spreading models of a Banach space are introduced, for every $k \in \mathbb{N}$. The definition, which is based on the k -sequences and plegma families, reveals a new class of spreading sequences associated to a Banach space. Most of the results of the classical theory are stated and proved in the higher order setting. Moreover, new phenomena like the universality of the class of the 2-spreading models of c_0 and the composition property are established. As consequence, a problem concerning the structure of the k -iterated spreading models is solved.

INTRODUCTION

The present work was motivated by a problem of E. Odell and Th. Schlumprecht concerning the structure of the k -iterated spreading models of the Banach spaces. Our attempt to answer the problem led to the k -spreading models which in turn are based on the k -sequences and plegma families. The aim of this paper is to introduce the above concepts and to develop a theory yielding, among others, a solution to the aforementioned problem.

Spreading models, invented by A. Brunel and L. Sucheston (c.f. [7]), possess a key role in the modern Banach space theory. Let us recall that a spreading model of a Banach space X is a spreading sequence¹ generated by a sequence of X . The spreading sequences have regular structure and the spreading models act as the tool for realizing that structure in the space X in an asymptotic manner. This together with the Brunel-Sucheston's discovery that every bounded sequence has a subsequence generating a spreading model determine the significance and importance of this concept. For a comprehensive presentation of the theory of the spreading models we refer the interested reader to the monograph of B. Beauzamy and J.-T. Lapresté (c.f. [5]).

Iteration is naturally applicable to spreading models. Thus one could define the 2-iterated spreading models of a Banach space X to be the spreading sequences which occur as spreading models of the spaces generated by spreading models of X . Further iteration yields the k -iterated spreading models of X , for every $k \in \mathbb{N}$. Iterated spreading models appeared in the literature shortly after Brunel-Sucheston's

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¹A sequence $(e_n)_n$ in a seminormed space $(E, \|\cdot\|_*)$ is called spreading if for every $n \in \mathbb{N}$, $k_1 < \dots < k_n$ in \mathbb{N} and $a_1, \dots, a_n \in \mathbb{R}$ we have that $\|\sum_{j=1}^n a_j e_j\|_* = \|\sum_{j=1}^n a_j e_{k_j}\|_*$.

In the literature the term “spreading model” usually indicates the space generated by the corresponding spreading sequence rather than the sequence itself. We have chosen to use the term for the spreading sequence and whenever we refer to ℓ^p or c_0 spreading model we shall mean that the spreading sequence is equivalent to the usual basis of the corresponding space.

invention. Indeed, B. Beauzamy and B. Maurey in [6], answering a problem of H.P. Rosenthal, showed that the class of the 2-iterated spreading models does not coincide with the corresponding one of the spreading models. In particular they constructed a Banach space admitting the usual basis of ℓ^1 as a 2-iterated spreading model and not as a spreading model.

E. Odell and Th. Schlumprecht in [17] asked whether or not every Banach space admits a k -iterated spreading model equivalent to the usual basis of ℓ^p , for some $1 \leq p < \infty$, or c_0 . Let us also point out that in the same paper they provided a reflexive space \mathfrak{X} with an unconditional basis such that no ℓ^p or c_0 is embedded into the space generated by any spreading model of the space. This remarkable result answered a long standing problem of the Banach space theory.

Our approach uses the k -spreading models which in many cases include the k -iterated ones. The k -spreading models are always spreading sequences $(e_n)_n$ in a seminormed space E . They are generated by k -sequences $(x_s)_{s \in [\mathbb{N}]^k}$, where $[\mathbb{N}]^k$ denotes the family of all k -subsets of \mathbb{N} . A critical ingredient in the definition is the plegma families $(s_i)_{i=1}^l$ of elements of $[\mathbb{N}]^k$, described as follows.

A finite sequence $(s_j)_{j=1}^l$ in $[\mathbb{N}]^k$ is a plegma family if its elements satisfy the following order relation: for every $1 \leq i \leq k$, $s_1(i) < \dots < s_l(i)$ and for every $1 \leq i < k$, $s_i(i) < s_1(i+1)$. The plegma families, as they are used in the definition, force a weaker asymptotic relation of the k -spreading models to the space X , as k increases. For $k = 1$, the plegma families coincide to the finite subsets of \mathbb{N} yielding that the new definition of the 1-spreading models recovers the classical one. For $k > 1$, the plegma families have a quite strict behavior which is described in the first section of the paper. Of independent interest is also Lemma 2 stated below.

The k -spreading models of a Banach space X are denoted by $\mathcal{SM}_k(X)$ and they define an increasing sequence. As the definition easily yields, the same holds for the k -iterated ones. Similarly to the classical case, for every bounded k -sequence $(x_s)_{s \in [\mathbb{N}]^k}$ there exists an infinite subset L of \mathbb{N} such that the k -subsequence $(x_s)_{s \in [L]^k}$ generates a k -spreading model.

The advantage of the k -spreading models is that, unlike the k -iterated ones, for $k \geq 2$, the space X determines directly their norm, through the k -sequences. Moreover, the k -spreading models have a transfinite extension yielding a hierarchy of ξ -spreading models for all $\xi < \omega_1$. The definition and the study of this hierarchy is more involved and will be presented elsewhere. We should also mention that L. Halbeisen and E. Odell (c.f. [10]) introduced the asymptotic models which share some common features with the 2-spreading models. The asymptotic models are associated to bounded 2-sequences $(x_s)_{s \in [\mathbb{N}]^2}$ and they are not necessarily spreading sequences.

The paper mainly concerns the definition and the study of the k -spreading models. Highlighting the results of the paper we should mention the universal property satisfied by the 2-spreading models of c_0 . More precisely, it is shown that every spreading sequence is isomorphically equivalent to some 2-spreading model of c_0 . As the spaces generated by k -iterated spreading models of c_0 are isomorphic to c_0 , the previous result shows that the k -spreading models do not coincide with the k -iterated ones. The composition property is also established. Roughly speaking, under some natural conditions, the d -spreading model of a k -spreading model of a Banach space X is a $(k + d)$ -spreading model of X . This result is used for showing that a special class of the k -iterated spreading models are actually k -spreading

models. We also extend to the higher order results of the spreading model theory. Among others we provide conditions for the k -sequences to generate unconditional spreading models and we study properties like non-distortion and duality of ℓ^1 and c_0 k -spreading models. Moreover we introduce the Cesàro summability for k -sequences and we prove the following that extends a classical theorem due to H.P. Rosenthal (c.f. [15, 19]).

Theorem 1. *Let X be a Banach space, $k \in \mathbb{N}$ and $(x_s)_{s \in [\mathbb{N}]^k}$ be a weakly relatively compact k -sequence in X , i.e. $\overline{\{x_s : s \in [\mathbb{N}]^k\}}^w$ is w -compact. Then there exists $M \in [\mathbb{N}]^\infty$ such that at least one of the following holds:*

- (1) *The subsequence $(x_s)_{s \in [M]^k}$ generates a k -spreading model equivalent to the usual basis of ℓ^1 .*
- (2) *There exists $x_0 \in X$ such that for every $L \in [M]^\infty$, $(x_s)_{s \in [L]^k}$ is k -Cesàro summable to x_0 .*

There are significant differences between the cases $k = 1$ and $k \geq 2$. First for $k = 1$ the two alternatives are exclusive which does not remain valid for $k \geq 2$. Second the proof for the case $k \geq 2$ uses the following density result concerning plegma families which is a consequence of the multidimensional Szemerédi's theorem due to H. Furstenberg and Y. Katznelson (c.f. [8]).

Lemma 2. *Let $\delta > 0$ and $k, l \in \mathbb{N}$. Then there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and every subset \mathcal{A} of the set of all k -subsets of $\{1, \dots, n\}$ of size at least $\delta \binom{n}{k}$, there exists a plegma l -tuple $(s_j)_{j=1}^l$ in \mathcal{A} .*

We close the paper with two examples. The first one is a Banach space similar to the aforementioned one of Odell-Schlumprecht. It is proved that no k -spreading model of the space is isomorphic to some ℓ^p , $1 \leq p < \infty$, or c_0 . The composition property, mentioned above, yields that the same holds for the k -iterated spreading models and thus the answer to the aforementioned Odell-Schlumprecht problem is a negative one. In the second example, for every $k \in \mathbb{N}$ we present a space \mathfrak{X}_{k+1} admitting the usual basis of ℓ^1 as a $(k+1)$ -spreading model while for every $d \leq k$, \mathfrak{X}_{k+1} does not admit ℓ^1 as a d -spreading model. As we have mentioned, the corresponding problem for k -iterated spreading models has been answered in [6] for $k+1 = 2$. It seems that for $k > 1$ this problem is still open. However, recently the $(k+1)$ -iterated spreading models have been separated by the k ones in [3]. The proofs in both examples make use of the results exhibited in the previous sections of the paper.

Notation. By $\mathbb{N} = \{1, 2, \dots\}$ we denote the set of all positive integers. We will use capital letters as L, M, N, \dots (resp. lower case letters as s, t, u, \dots) to denote infinite subsets (resp. finite subsets) of \mathbb{N} . For every infinite subset L of \mathbb{N} , the notation $[L]^\infty$ (resp. $[L]^{<\infty}$) stands for the set of all infinite (resp. finite) subsets of L . For every $s \in [\mathbb{N}]^{<\infty}$, by $|s|$ we denote the cardinality of s . For $L \in [\mathbb{N}]^\infty$ and $k \in \mathbb{N}$, $[L]^k$ (resp. $[L]^{\leq k}$) is the set of all $s \in [L]^{<\infty}$ with $|s| = k$ (resp. $|s| \leq k$). For every $s, t \in [\mathbb{N}]^{<\infty}$, we write $s < t$ if either at least one of them is the empty set, or $\max s < \min t$.

Throughout the paper we shall identify strictly increasing sequences in \mathbb{N} with their corresponding range, i.e. we view every strictly increasing sequence in \mathbb{N} as a subset of \mathbb{N} and conversely every subset of \mathbb{N} as the sequence resulting from the increasing ordering of its elements. Thus, for an infinite subset $L = \{l_1 < l_2 < \dots\}$

of \mathbb{N} and $i \in \mathbb{N}$, we set $L(i) = l_i$ and similarly, for a finite subset $s = \{n_1 < \dots < n_k\}$ of \mathbb{N} and for $1 \leq i \leq k$, we set $s(i) = n_i$. Also, for every $L, N \in [\mathbb{N}]^\infty$ and $s \in [\mathbb{N}]^{<\infty}$, we set $L(N) = \{L(N(i)) : i \in \mathbb{N}\}$ and $L(s) = \{L(s(i)) : 1 \leq i \leq |s|\}$. Similarly, for every $s \in [\mathbb{N}]^k$ and $F \subseteq \{1, \dots, k\}$, we set $s(F) = \{s(i) : i \in F\}$. Also for $1 \leq m \leq k$, we set $s|m = \{s(i) : 1 \leq i \leq m\}$.

For every $s, t \in [\mathbb{N}]^{<\infty}$, we write $s \sqsubseteq t$ (resp. $s \sqsubset t$) to denote that s is an initial (resp. proper initial) segment of t . Given two sequences $(s_j^1)_{j=1}^{l_1}$ and $(s_j^2)_{j=1}^{l_2}$ in $[\mathbb{N}]^{<\infty}$, by $(s_j^1)_{j=1}^{l_1} \hat{\ } (s_j^2)_{j=1}^{l_2}$, we denote their concatenation. Similarly for more than two sequences.

For a Banach space X with a Schauder basis $(e_n)_n$ and every $x \in X$, $x = \sum_n \lambda_n e_n$ we write $\text{supp}(x)$ to denote the support of x , i.e. $\text{supp}(x) = \{n \in \mathbb{N} : \lambda_n \neq 0\}$. If the support of x is finite and $E \subseteq \mathbb{N}$ then by $E(x)$, we denote the restriction of x to E , namely $E(x) = \sum_{n \in E} \lambda_n e_n$.

Two sequences $(x_n)_n$ and $(y_n)_n$, not necessarily in the same Banach space, will be called isometric (resp. equivalent) if (resp. there exists $0 < c \leq C$ such that) for every $n \in \mathbb{N}$ and $a_1, \dots, a_n \in \mathbb{R}$ we have that $\|\sum_{i=1}^n a_i x_i\| = \|\sum_{i=1}^n a_i y_i\|$ (resp. $c \|\sum_{i=1}^n a_i x_i\| \leq \|\sum_{i=1}^n a_i y_i\| \leq C \|\sum_{i=1}^n a_i x_i\|$). Generally concerning Banach space theory the notation and the terminology that we follow is the standard one (see [1] and [14]).

1. PLEGMA FAMILIES IN $[\mathbb{N}]^k$

As we have already mentioned, the basic ingredients of the definition of the k -spreading models are the k -sequences and the plegma families. In this section we introduce the plegma families as well as the related notions of the plegma paths and the plegma preserving maps.

1.1. Definition and basic properties. We start with the definition of the plegma families.

Definition 3. Let $k \in \mathbb{N}$ and $M \in [\mathbb{N}]^\infty$. A plegma family in $[M]^k$ is a finite sequence $(s_j)_{j=1}^l$ in $[M]^k$ satisfying the following properties.

- (i) For every $1 \leq i \leq k$, $s_1(i) < \dots < s_l(i)$.
- (ii) For every $1 \leq i < k$, $s_l(i) < s_1(i+1)$.

For each $l \in \mathbb{N}$, the set of all sequences $(s_j)_{j=1}^l$ which are plegma families in $[M]^k$ will be denoted by $Plm_l([M]^k)$. We also set $Plm([M]^k) = \bigcup_{l=1}^\infty Plm_l([M]^k)$.

Notice that for $l = 1$ and every $k \in \mathbb{N}$, we have $Plm_1([M]^k) = [M]^k$. Moreover, for $k = 1$ and every $l \in \mathbb{N}$, $Plm_l([M]^1) = [M]^l$. In the sequel the elements of $Plm_2([M]^k)$ will be called plegma pairs in $[M]^k$.

Remark 1. Although the notion of the plegma family is natural, it does not seem to have appeared in the literature. As it was pointed out to us by S. Todorčević, a concept that slightly reminds plegma pairs in $[\mathbb{N}]^3$ is given by E. Specker in [20].

In the next proposition we gather some useful properties of plegma families. The proof is straightforward.

Proposition 4. Let $k, l \in \mathbb{N}$, $M \in [\mathbb{N}]^\infty$ and $(s_j)_{j=1}^l$ be a finite sequence in $[M]^k$.

- (i) $(s_j)_{j=1}^l \in Plm_l([M]^k)$ if and only if there exists $F \in [M]^{kl}$ such that $s_j(i) = F((i-1)k + j)$, for every $1 \leq i \leq k$ and $1 \leq j \leq l$.

- (ii) If $(s_j)_{j=1}^l \in Plm_l([M]^k)$ then $(s_{j_p})_{p=1}^m \in Plm_m([M]^k)$, for every $1 \leq m \leq l$ and $1 \leq j_1 < \dots < j_m \leq l$.
- (iii) $(s_j)_{j=1}^l \in Plm_l([M]^k)$ if and only if (s_{j_1}, s_{j_2}) is a plegma pair in $[M]^k$, for every $1 \leq j_1 < j_2 \leq l$.
- (iv) If $(s_j)_{j=1}^l \in Plm_l([M]^k)$ then $(s_j(F))_{j=1}^l \in Plm_l([M]^{|F|})$, for every non empty $F \subseteq \{1, \dots, k\}$.

Theorem 5. Let M be an infinite subset of \mathbb{N} and $k, l \in \mathbb{N}$. Then for every finite partition $Plm_l([M]^k) = \bigcup_{j=1}^p P_j$, there exist $L \in [M]^\infty$ and $1 \leq j_0 \leq p$ such that $Plm_l([L]^k) \subseteq P_{j_0}$.

Proof. By Proposition 4 (i), we conclude that the map sending each plegma family $(s_j)_{j=1}^l$ in $[M]^k$ to its union $\bigcup_{j=1}^l s_j$ is a bijection from $Plm_l([M]^k)$ onto $[M]^{kl}$. Therefore the partition of $Plm_l([M]^k)$ induces a corresponding one to $[M]^{kl}$ and the conclusion easily follows by applying the Ramsey's theorem [18]. \square

1.2. Plegma paths in $[\mathbb{N}]^k$. In this subsection we introduce the definition of the plegma paths. As we shall see in the sequel, the plegma paths play important role in the development of the theory of k -spreading models.

Definition 6. Let $l, k \in \mathbb{N}$ and $M \in [\mathbb{N}]^\infty$. We will say that a finite sequence $(s_j)_{j=0}^l$ is a plegma path of length l from s_0 to s_l in $[M]^k$, if (s_{j-1}, s_j) is a plegma pair in $[M]^k$, for every $1 \leq j \leq l$.

Lemma 7. Let $k \in \mathbb{N}$ and $(s_j)_{j=0}^l$ be a plegma path in $[\mathbb{N}]^k$. If $s_0 < s_l$ then $l \geq k$.

Proof. Suppose on the contrary that $s_0 < s_l$ and $l < k$. Since (s_{j-1}, s_j) is a plegma pair in $[\mathbb{N}]^k$, we have $s_j(i_1) < s_{j-1}(i_2)$, for every $1 \leq j \leq l$ and $1 \leq i_1 < i_2 \leq k$. Hence, $s_l(1) < s_{l-1}(2) < s_{l-2}(3) < \dots < s_0(l+1) \leq s_0(k)$, which contradicts that $s_0 < s_l$. \square

Definition 8. Let $k \in \mathbb{N}$ and $M \in [\mathbb{N}]^\infty$. An $s \in [M]^k$ will be called *skipped* in M if for every $1 \leq i < k$ there exists $m \in M$ such that $s(i) < m < s(i+1)$. The set of all skipped $s \in [M]^k$ in M will be denoted by $[M]_{||}^k$.

Remark 2. Notice that for every $m \in \mathbb{N}$ and $s \in [M]_{||}^k$ there exists a plegma path $(s_j)_{j=0}^l$ in $[M]^k$ with $s_0 = s$.

Proposition 9. Let $k \in \mathbb{N}$ and $M \in [\mathbb{N}]^\infty$. Then for every $s, t \in [M]_{||}^k$ with $s < t$ there exists a plegma path of length k in $[M]^k$ from s to t . Moreover, every plegma path in $[\mathbb{N}]^k$ from s to t has length at least k .

Proof. Fix $s, t \in [M]_{||}^k$ with $s < t$. It is clear that we may choose $\tilde{s}, \tilde{t} \in [M]^{2k-1}$ such that $\tilde{s}(2i-1) = s(i)$ and similarly $\tilde{t}(2i-1) = t(i)$, for every $1 \leq i \leq k$. For every $0 \leq j \leq k$, we set

$$s_j = \{\tilde{s}(2i-1+j) : 1 \leq i \leq k-j\} \cup \{\tilde{t}(2i-1+k-j) : 1 \leq i \leq j\}$$

It is easy to check that $s_0 = s$, $s_k = t$ and $(s_j)_{j=0}^k$ is a plegma path in $[M]^k$. Moreover, by Lemma 7, every plegma path in $[\mathbb{N}]^k$ from s to t is of length at least k . Hence $(s_j)_{j=0}^k$ is a plegma path from s to t in $[M]^k$ with the least possible length and the proof is complete. \square

Remark 3. In terms of graph theory the above proposition states that in the directed graph with vertices the elements of $[\mathbb{N}]^k$ and edges the plegma pairs (s, t) in $[\mathbb{N}]^k$, the distance between two vertices s and t with $s < t$ is equal to k .

1.3. Plegma families and mappings.

Definition 10. Let $k_1, k_2 \in \mathbb{N}$, $M \in [\mathbb{N}]^\infty$ and $\varphi : [M]^{k_1} \rightarrow [\mathbb{N}]^{k_2}$. We will say that the map φ is plegma preserving from $[M]^{k_1}$ into $[\mathbb{N}]^{k_2}$ if for every plegma family $(s_j)_{j=1}^l$ in $[M]^{k_1}$, $(\varphi(s_j))_{j=1}^l$ is a plegma family in $[\mathbb{N}]^{k_2}$.

Remark 4. Let $k_1, k_2 \in \mathbb{N}$. If $k_1 < k_2$ then for every $M \in [\mathbb{N}]^\infty$ there exists a plegma preserving map from $[M]^{k_2}$ onto $[M]^{k_1}$. For instance, by Proposition 4, the map $s \rightarrow s|_{k_1}$ is plegma preserving from $[M]^{k_2}$ onto $[M]^{k_1}$.

In contrast to the above remark we have the following.

Theorem 11. Let $k_1, k_2 \in \mathbb{N}$. If $k_1 < k_2$ then for every $M \in [\mathbb{N}]^\infty$ and $\varphi : [M]^{k_1} \rightarrow [\mathbb{N}]^{k_2}$ there exists $L \in [M]^\infty$ such that for every plegma pair (s_1, s_2) in $[L]^{k_1}$ neither $(\phi(s_1), \phi(s_2))$ nor $(\phi(s_2), \phi(s_1))$ is a plegma pair in $[\mathbb{N}]^{k_2}$. In particular, there exists no $L \in [M]^\infty$ such that the map φ is plegma preserving from $[L]^{k_1}$ into $[\mathbb{N}]^{k_2}$.

Proof. Let $M \in [\mathbb{N}]^\infty$ and $\varphi : [M]^{k_1} \rightarrow [\mathbb{N}]^{k_2}$. We set P_1 (resp. P_2) to be the set of all $(s_1, s_2) \in Plm_2([M]^{k_1})$ such that $(\varphi(s_1), \varphi(s_2))$ (resp. $(\varphi(s_2), \varphi(s_1))$) is a plegma pair in $[\mathbb{N}]^{k_2}$ and $P_3 = Plm_2([M]^{k_1}) \setminus (P_1 \cup P_2)$. By Theorem 5 there exist $i \in \{1, 2, 3\}$ and $L \in [M]^\infty$ such that $Plm_2([L]^{k_1}) \subseteq P_i$. It remains to show that $i = 3$.

Indeed, assume that $i = 2$. By Remark 2 we may choose a plegma path $(s_j)_{j=0}^l$ in $[L]^k$ with $\min(\varphi(s_0)) < l$. For every $0 \leq j \leq l$, we set $n_j = \min(\varphi(s_j))$. Since $Plm_2([L]^{k_1}) \subseteq P_2$, we have that $(n_j)_{j=0}^l$ is a strictly decreasing sequence in \mathbb{N} with length $l + 1$. Since $n_0 < l$ this is impossible.

It remains to show that $i \neq 1$. Indeed, assume on the contrary. Then notice that φ transforms every plegma path in $[L]^{k_1}$ to a plegma path of equal length in $[\mathbb{N}]^{k_2}$. Using Remark 2, it is easy to see that we may choose $s < t$ in $[L]^{k_1}$ such that $\varphi(s) < \varphi(t)$ and $\varphi(s), \varphi(t) \in [\mathbb{N}]^{k_2}$. By Proposition 9 and Remark 3, we have that the distance of s, t is equal to k while that of $\varphi(s), \varphi(t)$ is equal to k_2 . But since s, t are joined by a plegma path of length k_1 and φ preserves plegma paths we have that the distance of $\varphi(s), \varphi(t)$ is at most k_1 . Hence $k_2 \leq k_1$, a contradiction. \square

Proposition 12. Let A be a set, $k \in \mathbb{N}$, $M \in [\mathbb{N}]^\infty$ and $\varphi : [M]^k \rightarrow A$. Then there exists $L \in [M]^\infty$ such that either the restriction of φ on $[L]^k$ is constant or for every plegma pair (s_1, s_2) in $[L]^k$, $\varphi(s_1) \neq \varphi(s_2)$.

Proof. By Theorem 5 there exists $N \in [M]^\infty$ such that exactly one of the following are satisfied.

- (i) For every plegma pair (s_1, s_2) in $[N]^k$, $\varphi(s_1) = \varphi(s_2)$.
- (ii) For every plegma pair (s_1, s_2) in $[N]^k$, $\varphi(s_1) \neq \varphi(s_2)$.

Therefore, it suffices to show that the first alternative implies that there exists $L \in [N]^\infty$ such that φ is constant on $[L]^k$. Indeed, let $s = (N(2), N(4), \dots, N(2k))$, $L = \{N(2n) : n \geq k + 1\}$ and $t \in [L]^k$. Observe that $s < t$ and $s, t \in [N]_n^k$ and therefore, by Proposition 9, there exists a plegma path $(s_j)_{j=0}^k$ of length k in $[N]^k$ with $s_0 = s$ and $s_k = t$. Assuming that (i) holds, we get that

$$\varphi(s) = \varphi(s_0) = \varphi(s_1) = \dots = \varphi(s_k) = \varphi(t)$$

Hence for every $t \in [L]^k$, $\varphi(t) = \varphi(s)$, i.e. φ is constant on $[L]^k$. \square

2. SPREADING SEQUENCES

We recall that a sequence $(e_n)_n$ in a seminormed linear space $(E, \|\cdot\|_*)$ is called *spreading* if it is isometric to any of its subsequences, i.e. for every $n \in \mathbb{N}$, $a_1, \dots, a_n \in \mathbb{R}$ and $k_1 < \dots < k_n$ in \mathbb{N} we have that $\|\sum_{j=1}^n a_j e_j\|_* = \|\sum_{j=1}^n a_j e_{k_j}\|_*$. In this section we will briefly discuss the norm properties of the spreading sequences. The interested reader can find a detailed analysis in the monographs [1] and [5].

The proof of the following result shares similar ideas with the one of Proposition I.1.B.2 in [5].

Proposition 13. *Let $(E, \|\cdot\|_*)$ be a seminormed linear space and $(e_n)_n$ be a spreading sequence in E . Then the following are equivalent.*

- (i) *There exist $n \in \mathbb{N}$ and $a_1, \dots, a_n \in \mathbb{R}$ not all zero, with $\|\sum_{i=1}^n a_i e_i\|_* = 0$.*
- (ii) *For every $n, m \in \mathbb{N}$, $\|e_n - e_m\|_* = 0$.*
- (iii) *For every $n \in \mathbb{N}$ and $a_1, \dots, a_n \in \mathbb{R}$, $\|\sum_{i=1}^n a_i e_i\|_* = |\sum_{i=1}^n a_i| \cdot \|e_1\|_*$.*

Spreading sequences in seminormed linear spaces satisfying (i)-(iii) of the above proposition will be called *trivial*. By (i) we have that if $(e_n)_n$ is non trivial, then $(e_n)_n$ is linearly independent and the restriction of the seminorm $\|\cdot\|_*$ to the linear subspace of E generated by $(e_n)_n$ is actually norm. Therefore, every non trivial spreading sequence generates a Banach space.

We classify the non trivial spreading sequences into the following three categories:

- (1) The *singular* spreading sequences, i.e. the non trivial spreading sequences which are not Schauder basic sequences.
- (2) the *unconditional* spreading sequences and
- (3) the *conditional Schauder basic* spreading sequences, i.e. the non trivial spreading sequences which are Schauder basic but not unconditional.

The next two results are restatements of Propositions I.1.4 and I.4.2 of [5] respectively.

Proposition 14. *Let $(e_n)_n$ be a non trivial spreading sequence. Then the following are equivalent.*

- (i) *$(e_n)_n$ is unconditional and not equivalent to the usual basis of ℓ^1 .*
- (ii) *$(e_n)_n$ is weakly null.*
- (iii) *$(e_n)_n$ is Cesàro summable to zero.*
- (iv) *$(e_n)_n$ is 1-unconditional and not equivalent to the usual basis of ℓ^1 .*

Proposition 15. *Let $(e_n)_n$ be a non trivial spreading sequence and E the Banach space generated by $(e_n)_n$. Then $(e_n)_n$ is singular if and only if $(e_n)_n$ is weakly convergent to a nonzero element $e \in E$.*

Remark 5. Let $(e_n)_n$ be a singular spreading sequence. By the above proposition, we have that $(e_n)_n$ is of the form $e_n = e'_n + e$, where e is nonzero and $(e'_n)_n$ is weakly null. This decomposition of $(e_n)_n$ as $e_n = e'_n + e$ will be called *the natural decomposition* of $(e_n)_n$. It is easy to check that $(e'_n)_n$ is non trivial, spreading and not equivalent to the usual basis of ℓ^1 . Hence by Proposition 15, $(e'_n)_n$ is unconditional, weakly null and Cesàro summable to zero. Moreover, if E and E' are the Banach spaces generated by the sequences $(e_n)_n$ and $(e'_n)_n$ respectively, then E, E' are isomorphic and $E = E' \oplus \langle e \rangle$.

Finally for the conditional Schauder basic spreading sequences we have the next characterization, which is a consequence of the above results and Rosenthal's ℓ^1 theorem [19].

Proposition 16. *Let $(e_n)_n$ be a spreading non trivial sequence and E be the Banach space generated by $(e_n)_n$. Then $(e_n)_n$ is a conditional Schauder basic sequence if and only if $(e_n)_n$ is non trivial weak-Cauchy.*

3. k -SEQUENCES AND k -SPREADING MODELS

In this section we present the definition of the k -sequences and we introduce the notion of the k -spreading models, for all $k \in \mathbb{N}$. As we will see, for $k = 1$, the definition coincides with the classical one of A. Brunel and L. Sucheston [7].

3.1. Definitions and basic properties. We start with the definition of the k -sequences.

Definition 17. *Let $k \in \mathbb{N}$ and X be a non empty set. A k -sequence in X is a map $\varphi : [\mathbb{N}]^k \rightarrow X$. A k -subsequence in X is a map of the form $\varphi : [M]^k \rightarrow X$, where $M \in [\mathbb{N}]^\infty$.*

A k -sequence $\varphi : [\mathbb{N}]^k \rightarrow X$ will be usually denoted by $(x_s)_{s \in [\mathbb{N}]^k}$, where $x_s = \varphi(s)$, $s \in [\mathbb{N}]^k$. Similarly, the notation $(x_s)_{s \in [M]^k}$ stands for the k -subsequences $\varphi : [M]^k \rightarrow X$.

Definition 18. *Let X be a Banach space, $k \in \mathbb{N}$, $(x_s)_{s \in [\mathbb{N}]^k}$ be a k -sequence in X and $(E, \|\cdot\|_*)$ be an infinite dimensional seminormed linear space with Hamel basis $(e_n)_n$. Also let $M \in [\mathbb{N}]^\infty$ and $(\delta_n)_n$ be a null sequence of positive reals. We will say that the k -subsequence $(x_s)_{s \in [M]^k}$ generates $(e_n)_n$ as a spreading model as a k -spreading model (with respect to $(\delta_n)_n$), if the following is satisfied.*

For every $m, l \in \mathbb{N}$, with $m \leq l$, every $(s_j)_{j=1}^m \in \text{Plm}_m([M]^k)$ with $s_1(1) \geq M(l)$ and every choice of $a_1, \dots, a_m \in [-1, 1]$, we have

$$(1) \quad \left| \left\| \sum_{j=1}^m a_j x_{s_j} \right\| - \left\| \sum_{j=1}^m a_j e_j \right\|_* \right| \leq \delta_l$$

Since $\text{Plm}([\mathbb{N}]^1) = [\mathbb{N}]^{<\infty}$, it is clear that for $k = 1$, Definition 18 coincides with the classical definition of a spreading model of an ordinary sequence $(x_n)_n$ in a Banach space X . Thus the 1-spreading models are the usual ones. Moreover, it is easy to see that for every $k \in \mathbb{N}$, every k -spreading model $(e_n)_n$ is a spreading sequence.

Let's point out here that there exist k -sequences in Banach spaces which generate k -spreading models which are trivial spreading sequences, in other words (see Proposition 13), $\|\cdot\|_*$ is not a norm. For instance, this occurs for every constant k -sequence $(x_s)_{s \in [\mathbb{N}]^k}$. We should also point out that even if $(e_n)_n$ is non trivial, it is not necessarily a Schauder basic sequence. More information on this issue are contained in Section 6.

In the next proposition we state some stability properties of the k -spreading models. The proof is straightforward.

Proposition 19. *Let $k \in \mathbb{N}$, $(x_s)_{s \in [\mathbb{N}]^k}$ be a k -sequence in a Banach space X , $M \in [\mathbb{N}]^\infty$ and $(\delta_n)_n$ be a null sequence of positive reals. If $(x_s)_{s \in [M]^k}$ generates a*

sequence $(e_n)_n$ as a k -spreading model with respect to $(\delta_n)_n$ then the following are satisfied.

- (i) For every $L \in [M]^\infty$, $(x_s)_{s \in [L]^k}$ generates $(e_n)_n$ as a k -spreading model with respect to $(\delta_n)_n$.
- (ii) For every null sequence $(\delta'_n)_n$ of positive reals there exists $M' \in [M]^\infty$ such that $(x_s)_{s \in [M']^k}$ generates $(e_n)_n$ as a k -spreading model with respect to $(\delta'_n)_n$.
- (iii) The k -sequence $(y_s)_{s \in [\mathbb{N}]^k}$, defined by $y_s = x_{M(s)}$, $s \in [\mathbb{N}]^k$, generates $(e_n)_n$ as a k -spreading model with respect to $(\delta_n)_n$.

Let us also notice that for $k = 1$ the assertion that (1) holds for all $m \leq l$ is redundant. This is not the case for $k \geq 2$, since a plegma family in $[\mathbb{N}]^k$ is not always a subsequence of a larger one. However, the next lemma shows that we may bypass this extra condition by passing to a sparse infinite subset of \mathbb{N} .

Lemma 20. *Let $k \in \mathbb{N}$, $(x_s)_{s \in [\mathbb{N}]^k}$ be a k -sequence in a Banach space X , $L \in [\mathbb{N}]^\infty$, $(E, \|\cdot\|_*)$ be an infinite dimensional seminormed linear space with Hamel basis $(e_n)_n$ and $(\delta_n)_n$ be a null sequence of positive reals such that*

$$(2) \quad \left| \left\| \sum_{j=1}^l a_j x_{t_j} \right\| - \left\| \sum_{j=1}^l a_j e_j \right\|_* \right| \leq \delta_l$$

for every $l \in \mathbb{N}$, every $(t_j)_{j=1}^l \in \text{Plm}_l([L]^k)$ with $t_1(1) \geq L(l)$ and every choice of $a_1, \dots, a_l \in [-1, 1]$. Then there exists $M \in [L]^\infty$ such that $(x_s)_{s \in [M]^k}$ generates $(e_n)_n$ as a k -spreading model with respect to $(\delta_n)_n$.

Proof. We choose $M \in [L]^\infty$ such that for every $l \in \mathbb{N}$ there exist at least $l - 1$ elements of L between $M(l)$ and $M(l + 1)$. Then notice that for every $m, l \in \mathbb{N}$ with $m \leq l$ and every $(s_j)_{j=1}^m \in \text{Plm}_m([M]^k)$ with $s_j(1) \geq M(l)$, there exists $(t_j)_{j=1}^l \in \text{Plm}_l([L]^k)$ with $s_j = t_j$ for all $1 \leq j \leq m$. This observation and (2) easily yield that for every $m, l \in \mathbb{N}$, with $m \leq l$, every $(s_j)_{j=1}^m \in \text{Plm}_m([M]^k)$ with $s_1(1) \geq M(l)$ and every choice of $a_1, \dots, a_m \in [-1, 1]$, we have

$$(3) \quad \left| \left\| \sum_{j=1}^m a_j x_{s_j} \right\| - \left\| \sum_{j=1}^m a_j e_j \right\|_* \right| \leq \delta_l$$

and the proof is complete. \square

3.2. Existence of k -spreading models. In this subsection we will show that every bounded k -sequence in a Banach space X contains a k -subsequence which generates a k -spreading model. The proof follows similar lines with the corresponding one of the classical spreading models.

For $k \in \mathbb{N}$ and a k -sequence $(x_s)_{s \in [\mathbb{N}]^k}$ in a Banach space X , we will say that $(x_s)_{s \in [\mathbb{N}]^k}$ admits $(e_n)_n$ as a k -spreading model (or $(e_n)_n$ is a k -spreading model of $(x_s)_{s \in [\mathbb{N}]^k}$) if there exists $M \in [\mathbb{N}]^\infty$ such that the subsequence $(x_s)_{s \in [M]^k}$ generates $(e_n)_n$ as a k -spreading model. A k -sequence $(x_s)_{s \in [\mathbb{N}]^k}$ in X will be called *bounded* (resp. *seminormalized*) if there exists $C > 0$ (resp. $0 < c \leq C$) such that $\|x_s\| \leq C$ (resp. $c \leq \|x_s\| \leq C$), for every $s \in [\mathbb{N}]^k$.

Theorem 21. *For all $k \in \mathbb{N}$, every bounded k -sequence in a Banach space X admits a k -spreading model.*

Proof. Let X be a Banach space and $k \in \mathbb{N}$, $(x_s)_{s \in [\mathbb{N}]^k}$ be a bounded k -sequence in X . We divide the proof into four steps.

Step 1. Let $l \in \mathbb{N}$, $N \in [\mathbb{N}]^\infty$ and $\delta > 0$. Then there exists $L \in [N]^\infty$ such that

$$\left| \left\| \sum_{j=1}^l a_j x_{t_j} \right\| - \left\| \sum_{j=1}^l a_j x_{s_j} \right\| \right| \leq \delta$$

for every $(t_j)_{j=1}^l, (s_j)_{j=1}^l \in Plm_l([L]^k)$ and $a_1, \dots, a_l \in [-1, 1]$.

Proof of Step 1: Let $(\mathbf{a}_i)_{i=1}^{n_0}$ be a $\frac{\delta}{3l}$ -net of the unit ball of $(\mathbb{R}^l, \|\cdot\|_\infty)$. We set $N_0 = N$. By a finite induction on $1 \leq i \leq n_0$, we construct a decreasing sequence $N_0 \supseteq N_1 \supseteq \dots \supseteq N_{n_0}$ as follows. Suppose that N_0, \dots, N_{i-1} have been constructed. Let $\mathbf{a}_i = (a_j^i)_{j=1}^l$ and $g_i : Plm_l([N_{i-1}]^k) \rightarrow [0, lC]$ defined by $g_i((s_j)_{j=1}^l) = \left\| \sum_{j=1}^l a_j^i x_{s_j} \right\|$. We partition the interval $[0, lC]$ into disjoint intervals of length $\frac{\delta}{3}$ and applying Theorem 5 we find $N_i \in [N_{i-1}]^\infty$ such that for every $(t_j)_{j=1}^l, (s_j)_{j=1}^l \in Plm_l([N_i]^k)$, we have $|g_i((t_j)_{j=1}^l) - g_i((s_j)_{j=1}^l)| < \frac{\delta}{3}$. Proceeding in this way we conclude that for every $(s_j)_{j=1}^l, (t_j)_{j=1}^l \in Plm_l([N_{n_0}]^k)$ and $1 \leq i \leq n_0$, we have that $\left| \left\| \sum_{j=1}^l a_j^i x_{t_j} \right\| - \left\| \sum_{j=1}^l a_j^i x_{s_j} \right\| \right| \leq \frac{\delta}{3}$. Since $(\mathbf{a}_i)_{i=1}^{n_0}$ is a $\frac{\delta}{3}$ -net of the unit ball of $(\mathbb{R}^l, \|\cdot\|_\infty)$ it is easy to see that $L = N_{n_0}$ is as desired. \square

Step 2. Let $(\delta_n)_n$ be a null sequence of positive real numbers. Then there exists $M \in [\mathbb{N}]^\infty$ such that for every $m \leq l$, every $(t_j)_{j=1}^m, (s_j)_{j=1}^m \in Plm_m([M]^k)$ with $s_1(1), t_1(1) \geq M(l)$ and $a_1, \dots, a_m \in [-1, 1]$, we have

$$(4) \quad \left| \left\| \sum_{j=1}^m a_j x_{t_j} \right\| - \left\| \sum_{j=1}^m a_j x_{s_j} \right\| \right| \leq \delta_l$$

Proof of Step 2: By Step 1 and a standard diagonalization we easily obtain an $L \in [\mathbb{N}]^\infty$ satisfying $\left| \left\| \sum_{j=1}^l a_j x_{t_j} \right\| - \left\| \sum_{j=1}^l a_j x_{s_j} \right\| \right| \leq \delta_l$, for every $l \in \mathbb{N}$, every $(t_j)_{j=1}^l, (s_j)_{j=1}^l \in Plm_l([L]^k)$ with $s_1(1), t_1(1) \geq L(l)$ and $a_1, \dots, a_l \in [-1, 1]$. By Lemma 20, there exists $M \in [L]^\infty$ satisfying (4). \square

Step 3. Let $M \in [\mathbb{N}]^\infty$ be the resulting from Step 2 infinite subset of \mathbb{N} . Also let $l \in \mathbb{N}$ and $a_1, \dots, a_l \in \mathbb{R}$. Then for every sequence $((s_j^n)_{j=1}^l)_n$ with $(s_j^n)_{j=1}^l \in Plm_l([M]^k)$, for all $n \in \mathbb{N}$ and $\lim s_1^n(1) = +\infty$, the sequence $(\left\| \sum_{j=1}^l a_j x_{s_j^n} \right\|)_n$ is a Cauchy sequence in $[0, +\infty)$. Moreover, $\lim_n \left\| \sum_{j=1}^l a_j x_{s_j^n} \right\|$ is independent from the choice of the sequence $((s_j^n)_{j=1}^l)_n$.

Proof of Step 3: It is straightforward by Step 2. \square

Step 4. Let $(e_n)_n$ be the natural Hamel basis of $c_{00}(\mathbb{N})$. For every $l \in \mathbb{N}$ and $a_1, \dots, a_l \in \mathbb{R}$, we define

$$\left\| \sum_{j=1}^l a_j e_j \right\|_* = \lim_n \left\| \sum_{j=1}^l a_j x_{s_j^n} \right\|$$

where for every $n \in \mathbb{N}$, $(s_j^n)_{j=1}^l \in Plm_l([M]^k)$ and $\lim s_1^n(1) = +\infty$. Then $\|\cdot\|_*$ is a seminorm on $c_{00}(\mathbb{N})$ under which the natural Hamel basis $(e_n)_n$ is a spreading

sequence. Moreover for all $m \leq l$, $a_1, \dots, a_m \in [-1, 1]$ and $(s_j)_{j=1}^m \in Plm_m([M]^k)$ with $s_1(1) \geq M(l)$, we have $\left| \left\| \sum_{j=1}^m a_j x_{s_j} \right\| - \left\| \sum_{j=1}^m a_j e_j \right\|_* \right| \leq \delta_l$.

Proof of Step 4: It follows easily by Steps 2 and 3. \square

By Step 4, we have that $(x_s)_{s \in [M]^k}$ generates $(e_n)_n$ as a k -spreading model and the proof is complete. \square

3.3. The increasing hierarchy of k -spreading models. In this subsection we will show that the k -spreading models of a Banach space X form an increasing hierarchy.

We start with the following lemma which is an easy consequence of Remark 4.

Lemma 22. *Let $k_1, k_2 \in \mathbb{N}$ with $1 \leq k_1 < k_2$. Let X be a Banach space and $(w_t)_{t \in [\mathbb{N}]^{k_1}}$ be a k_1 -sequence in X . Let $(x_s)_{s \in [\mathbb{N}]^{k_2}}$ be the k_2 -sequence in X defined by $x_s = w_{s|_{k_1}}$, for every $s \in [\mathbb{N}]^{k_2}$. Then $(w_t)_{t \in [\mathbb{N}]^{k_1}}$ and $(x_s)_{s \in [\mathbb{N}]^{k_2}}$ admit the same k -spreading models.*

For a subset A of X we will say that A admits $(e_n)_n$ as a k -spreading model (or $(e_n)_n$ is a k -spreading model of A) if there exists a k -sequence $(x_s)_{s \in [\mathbb{N}]^k}$ in A which admits $(e_n)_n$ as a k -spreading model.

Notation 1. Let X be a Banach space, $A \subseteq X$ and $k \in \mathbb{N}$. The set of all k -spreading models of A will be denoted by $\mathcal{SM}_k(A)$.

By Lemma 22, we easily obtain the following.

Corollary 23. *Let X be a Banach space and $A \subseteq X$. Then for all $k_1, k_2 \in \mathbb{N}$ with $k_1 < k_2$, we have $\mathcal{SM}_{k_1}(A) \subseteq \mathcal{SM}_{k_2}(A)$,*

In Section 12, for each $k \in \mathbb{N}$, we construct a Banach space \mathfrak{X}_{k+1} such that $\mathcal{SM}_k(\mathfrak{X}_{k+1}) \subsetneq \mathcal{SM}_{k+1}(\mathfrak{X}_{k+1})$. Here, we present a much simpler example of a space X and a proper subset A of X satisfying $\mathcal{SM}_k(A) \subsetneq \mathcal{SM}_{k+1}(A)$.

Example 1. Let $(e_n)_n$ be a normalized spreading and 1-unconditional sequence in a Banach space $(E, \|\cdot\|)$ which is not equivalent to the usual basis of c_0 . Let $k \in \mathbb{N}$ and $(x_s)_{s \in [\mathbb{N}]^{k+1}}$ be the natural Hamel basis of $c_{00}([\mathbb{N}]^{k+1})$. For $x \in c_{00}([\mathbb{N}]^{k+1})$ we define

$$\|x\|_{k+1} = \sup \left\{ \left\| \sum_{i=1}^l x(s_i) e_i \right\| : l \in \mathbb{N}, (s_i)_{i=1}^l \in Plm_l([\mathbb{N}]^{k+1}) \text{ and } s_1(1) \geq l \right\}$$

We set $X = \overline{(c_{00}([\mathbb{N}]^{k+1}), \|\cdot\|_{k+1})}$ and $A = \{x_s : s \in [\mathbb{N}]^{k+1}\}$. It is easy to see that the sequence $(e_n)_n$ is generated by $(x_s)_{s \in [\mathbb{N}]^{k+1}}$ as a $(k+1)$ -spreading model and thus it belongs to $\mathcal{SM}_{k+1}(A)$. We shall show that for every $(\tilde{e}_n)_n \in \mathcal{SM}_k(A)$, either $(\tilde{e}_n)_n$ is a trivial spreading sequence or it is isometric to the usual basis of c_0 . Therefore, there is no sequence in $\mathcal{SM}_k(A)$ equivalent to $(e_n)_n$.

Indeed, let $(\tilde{e}_n)_n \in \mathcal{SM}_k(A)$. By Proposition 19, we may assume that there exists a k -sequence in A , $(y_t)_{t \in [\mathbb{N}]^k}$ which generates $(\tilde{e}_n)_n$ as a k -spreading model. Let $\varphi : [\mathbb{N}]^k \rightarrow [\mathbb{N}]^{k+1}$ such that $y_t = x_{\varphi(t)}$, for all $t \in [\mathbb{N}]^k$. By Proposition 12, there exists $M \in [\mathbb{N}]^\infty$ such that either φ is constant on $[M]^k$ or for every plegma pair (t_1, t_2) in $[M]^k$, $\varphi(t_1) \neq \varphi(t_2)$. By Proposition 19, we have that $(y_t)_{t \in [M]^k}$ also generates $(\tilde{e}_n)_n$ as a k -spreading model.

If φ is constant on $[M]^k$ then $(\tilde{e}_n)_n$ is a trivial sequence. Otherwise, by Theorem 11, there exists $L \in [M]^\infty$ such that for every plegma pair (t_1, t_2) in $[L]^k$ neither $(\varphi(t_1), \varphi(t_2))$, nor $(\varphi(t_2), \varphi(t_1))$ is a plegma pair in $[\mathbb{N}]^{k+1}$. Therefore, for every $(t_j)_{j=1}^m \in Plm([L]^k)$ and $(s_j)_{j=1}^l \in Plm([\mathbb{N}]^{k+1})$ there is at most one $j \in \{1, \dots, m\}$ and at most one $i \in \{1, \dots, l\}$ with $\varphi(t_j) = s_i$. This observation and the definition of the norm $\|\cdot\|_{k+1}$, easily implies that

$$(5) \quad \left\| \sum_{j=1}^m a_j y_{t_j} \right\|_{k+1} = \left\| \sum_{j=1}^m a_j x_{\varphi(t_j)} \right\|_{k+1} = \max_{1 \leq j \leq m} |a_j|$$

for all $m \in \mathbb{N}$, $a_1, \dots, a_m \in \mathbb{R}$ and $(t_j)_{j=1}^m \in Plm([L]^k)$. Since $L \in [M]^\infty$, we have that $(\tilde{e}_n)_n$ is generated by $(y_t)_{t \in [L]^k}$ and by (5), the sequence $(\tilde{e}_n)_n$ is isometric to the usual basis of c_0 .

4. TOPOLOGICAL PROPERTIES OF k -SEQUENCES

This section is devoted to the study of the k -sequences in a topological space. We define the convergence of the k -sequences in a topological space and we introduce the notion of the subordinated k -sequences.

4.1. Convergence of k -sequences in topological spaces. We start with the following natural extension of the notion of convergence of sequences in topological spaces.

Definition 24. Let (X, \mathcal{T}) be a topological space, $k \in \mathbb{N}$ and $(x_s)_{s \in [\mathbb{N}]^k}$ be a k -sequence in X . Also let $M \in [\mathbb{N}]^\infty$ and $x_0 \in X$. We will say that $(x_s)_{s \in [M]^k}$ converges to x_0 if for every $U \in \mathcal{T}$ with $x_0 \in U$ there exists $m \in \mathbb{N}$ such that for every $s \in [M]^k$ with $s(1) \geq M(m)$ we have that $x_s \in U$.

It is straightforward that if a k -subsequence $(x_s)_{s \in [M]^k}$ in a topological space is convergent to some $x_0 \in X$, then every further k -subsequence of $(x_s)_{s \in [M]^k}$ is also convergent to x_0 . Moreover, every continuous map between two topological spaces preserves the convergence of k -sequences, i.e. if $\phi : (X_1, \mathcal{T}_1) \rightarrow (X_2, \mathcal{T}_2)$ is continuous and $(x_s)_{s \in [M]^k}$ converges to $x_0 \in X_1$, then $(\phi(x_s))_{s \in [M]^k}$ converges to $\phi(x_0) \in X_2$.

However, for $k \geq 2$, there are some differences with the ordinary convergent sequences in topological spaces. For instance it is easy to see that for $k \geq 2$, the convergence of a k -sequence $(x_s)_{s \in [M]^k}$ to some $x_0 \in X$, does not in general imply that the set $\{x_s : s \in [M]^k\}$ is relatively compact.

4.2. Subordinated k -sequences. In this subsection we introduce the definition of the subordinated k -sequences in a topological space. First, recall that the powerset of \mathbb{N} is naturally identified with $\{0, 1\}^\mathbb{N}$. In this way, for all $k \in \mathbb{N}$ and $M \in [\mathbb{N}]^\infty$, the set $[M]^{\leq k}$ becomes a compact metric space containing $[M]^k$ as a dense subspace. Moreover, notice that an element $s \in [M]^{\leq k}$ is isolated in $[M]^{\leq k}$ if and only if $s \in [M]^k$.

Definition 25. Let (X, \mathcal{T}) be a topological space, $k \in \mathbb{N}$, $(x_s)_{s \in [\mathbb{N}]^k}$ be a k -sequence in X and $M \in [\mathbb{N}]^\infty$. We say that $(x_s)_{s \in [M]^k}$ is subordinated (with respect to (X, \mathcal{T})) if there exists a continuous map $\hat{\varphi} : [M]^{\leq k} \rightarrow (X, \mathcal{T})$ such that $\hat{\varphi}(s) = x_s$, for all $s \in [M]^k$.

Remark 6. If $(x_s)_{s \in [M]^k}$ is subordinated, then there exists a unique continuous map $\widehat{\varphi} : [M]^{\leq k} \rightarrow (X, \mathcal{T})$ witnessing this. Indeed, this is a consequence of the fact that $[M]^k$ is dense in $[M]^{\leq k}$. Also, $\overline{\{x_s : s \in [M]^k\}} = \widehat{\varphi}([M]^{\leq k})$, where $\overline{\{x_s : s \in [M]^k\}}$ is the closure of $\{x_s : s \in [M]^k\}$ in X with respect to \mathcal{T} . Therefore, $\overline{\{x_s : s \in [M]^k\}}$ is a countable compact metrizable subspace of (X, \mathcal{T}) with Cantor-Bendixson index at most $k + 1$. Also notice that if $(x_s)_{s \in [M]^k}$ is subordinated then $(x_s)_{s \in [L]^k}$ is also subordinated, for every $L \in [M]^\infty$.

Proposition 26. Let (X, \mathcal{T}) be a topological space, $k \in \mathbb{N}$, $(x_s)_{s \in [\mathbb{N}]^k}$ be a k -sequence in X and $M \in [\mathbb{N}]^\infty$. Suppose that $(x_s)_{s \in [M]^k}$ is subordinated and let $\widehat{\varphi} : [M]^{\leq k} \rightarrow (X, \mathcal{T})$ be the continuous map witnessing this. Then $(x_s)_{s \in [M]^k}$ is convergent to $\widehat{\varphi}(\emptyset)$.

Proof. Let $(y_s)_{s \in [M]^k}$ be the k -sequence in $[M]^k$, with $y_s = s$, for all $s \in [M]^k$. Notice that $(y_s)_{s \in [M]^k}$ converges to the empty set and since $\widehat{\varphi} : [M]^{\leq k} \rightarrow (X, \mathcal{T})$ is continuous, we have that $(\widehat{\varphi}(y_s))_{s \in [M]^k}$ converges to $\widehat{\varphi}(\emptyset)$. Since $\widehat{\varphi}(y_s) = \widehat{\varphi}(s) = x_s$, for all $s \in [M]^k$, we conclude that $(x_s)_{s \in [M]^k}$ is convergent to $\widehat{\varphi}(\emptyset)$. \square

Proposition 27. Let (X, \mathcal{T}) be a topological space, $k \in \mathbb{N}$ and $(x_s)_{s \in [\mathbb{N}]^k}$ be a k -sequence in X . Then for every $N \in [\mathbb{N}]^\infty$ such that $\overline{\{x_s : s \in [N]^k\}}$ is a compact metrizable subspace of (X, \mathcal{T}) there exists $M \in [N]^\infty$ such that $(x_s)_{s \in [M]^k}$ is subordinated.

Proof. The proposition obviously holds for $k = 1$, since in this case, subordinated and convergent sequences coincide. We proceed by induction on $k \in \mathbb{N}$. Assume that Proposition 27 holds for some $k \in \mathbb{N}$ and let $(x_s)_{s \in [N]^{k+1}}$ be a $(k+1)$ -sequence in X . Let $N \in [\mathbb{N}]^\infty$ such that $\overline{\{x_s : s \in [N]^{k+1}\}}$ is a compact metrizable subspace of (X, \mathcal{T}) . We also fix a compatible metric d of $\overline{\{x_s : s \in [N]^{k+1}\}}$.

Inductively we choose a strictly increasing sequence $(l_n)_n$ in \mathbb{N} , a decreasing sequence $(L_n)_n$ of infinite subsets of N and a k -sequence $(x_s)_{s \in [L]^k}$ in X , where $L = \{l_n : n \in \mathbb{N}\}$ such that for every $n \in \mathbb{N}$, the following are satisfied.

- (i) $l_n < \min L_n$.
- (ii) For every $l \in L_n$ and every $t \in [\{l_1, \dots, l_n\}]^k$, $(x_{t \cup \{l\}})_{l \in L_n} \rightarrow x_t$ and in addition if $\max t = l_n$, then $d(x_{t \cup \{l\}}, x_t) < \frac{1}{n}$.

We omit the construction since it is straightforward. By the inductive assumption there exists $M \in [L]^\infty$ such that $(x_t)_{t \in [M]^k}$ is subordinated. If $\widehat{\psi} : [M]^{\leq k} \rightarrow X$ is the continuous map witnessing this then we extend $\widehat{\psi}$ to the map $\widehat{\varphi} : [M]^{\leq k+1} \rightarrow X$, by setting $\widehat{\varphi}(s) = x_s$, for every $s \in [M]^{k+1}$. Using condition (ii), we easily show that $\widehat{\varphi}$ is continuous and therefore $(x_s)_{s \in [M]^{k+1}}$ is subordinated. \square

Remark 7. By Propositions 26 and 27, we have that every k -sequence in a compact metrizable space contains a convergent k -subsequence.

5. WEAKLY RELATIVELY COMPACT k -SEQUENCES IN BANACH SPACES

It is well known that for every sequence $(x_n)_n$ in a weakly compact subset of a Banach space X there exists $M \in \mathbb{N}$ such that the subsequence $(x_n)_{n \in M}$ is weakly convergent to some $x_0 \in X$. Moreover, if in addition X has a Schauder basis then we may pass to a further subsequence $(x_n)_{n \in L}$ which is approximated by a

sequence of the form $(\tilde{x}_n)_{n \in L}$ such that $(\tilde{x}_n)_{n \in L}$ also weakly converges to x_0 and $(\tilde{x}_n - x_0)_{n \in L}$ is a block sequence of X . The main aim of this section is to show that, for every $k \geq 2$, the k -sequences in Banach spaces satisfy similar properties.

Definition 28. A k -sequence $(x_s)_{s \in [\mathbb{N}]^k}$, of a Banach space X will be called weakly relatively compact if $\overline{\{x_s : s \in [\mathbb{N}]^k\}}^w$ is a weakly compact subset of X .

Since the weak topology on every separable weakly compact subset of a Banach space is metrizable, by Propositions 26 and 27 we have the following.

Proposition 29. Let X be a Banach space and $k \in \mathbb{N}$. Then we have the following.

- (i) Every subordinated k -sequence in (X, w) is weakly convergent.
- (ii) Every weakly relatively compact k -sequence in X contains a subordinated k -subsequence.

To describe the regularity properties of weakly relatively compact k -sequences in a Banach space X with Schauder basis we will need the next two definitions. The first is a natural extension of the notion of block (resp. disjointly supported) sequences of X .

Definition 30. Let X be a Banach space with a Schauder basis and $k \in \mathbb{N}$. Let also $(x_s)_{s \in [\mathbb{N}]^k}$ be a k -sequence in X and $M \in [\mathbb{N}]^\infty$. We will say that the k -subsequence $(x_s)_{s \in [M]^k}$ is plegma block (resp. plegma disjointly supported) if for all plegma pairs (s_1, s_2) in $[M]^k$ we have $\text{supp}(x_{s_1}) < \text{supp}(x_{s_2})$ (resp. $\text{supp}(x_{s_1}) \cap \text{supp}(x_{s_2}) = \emptyset$).

Definition 31. Let X a Banach space with a Schauder basis, $k \in \mathbb{N}$ and $(x_s)_{s \in [\mathbb{N}]^k}$ be a k -sequence in X . Also let $L \in [\mathbb{N}]^\infty$ and $(y_t)_{t \in [L]^{\leq k}}$ be a family of vectors in X . We will say that $(y_t)_{t \in [L]^{\leq k}}$ is a canonical tree decomposition of $(x_s)_{s \in [L]^k}$ (or $(x_s)_{s \in [L]^k}$ admits $(y_t)_{t \in [L]^{\leq k}}$ as a canonical tree decomposition) if the following are satisfied.

- (i) For every $s \in [L]^k$, $x_s = \sum_{j=0}^k y_{s|j} = y_0 + \sum_{j=1}^k y_{s|j}$.
- (ii) For every $t \in [L]^{\leq k} \setminus \{\emptyset\}$, $\text{supp}(y_t)$ is finite.
- (iii) For every $s \in [L]^k$ and $1 \leq j_1 < j_2 \leq k$, $\text{supp}(y_{s|j_1}) < \text{supp}(y_{s|j_2})$.
- (iv) For every $(s_1, s_2) \in \text{Plm}_2([L]^k)$ and $1 \leq j_1 \leq j_2 \leq k$, we have
$$\text{supp}(y_{s_1|j_1}) < \text{supp}(y_{s_2|j_2})$$
- (v) For every $(s_1, s_2) \in \text{Plm}_2([L]^k)$ and $1 \leq j_1 < j_2 \leq k$, we have
$$\text{supp}(y_{s_2|j_1}) < \text{supp}(y_{s_1|j_2})$$

The next proposition gathers some basic properties of the k -sequences which admit canonical tree decomposition. Its proof is straightforward.

Proposition 32. Let X a Banach space with a Schauder basis, $k \in \mathbb{N}$, $(x_s)_{s \in [\mathbb{N}]^k}$ be a k -sequence in X and $L \in [\mathbb{N}]^\infty$. Assume that $(x_s)_{s \in [L]^k}$ admits $(y_t)_{t \in [L]^{\leq k}}$ as a canonical tree decomposition. Then the following are satisfied.

- (i) For every $N \in [L]^\infty$, the k -subsequence $(x_s)_{s \in [N]^k}$ admits $(y_t)_{t \in [N]^{\leq k}}$ as a canonical tree decomposition.
- (ii) For every $s \in [L]^k$, the sequence $(y_{s|j})_{j=1}^k$ is a block sequence in X .
- (iii) For every $1 \leq j \leq k$, the sequence $(y_{s|j})_{s \in [L]^k}$ is a plegma block k -sequence in X .

- (iv) Setting $x'_s = x_s - y_0$, for all $s \in [L]^k$, $y'_0 = 0$ and $y'_t = y_t$, for all $t \in [L]^{\leq k}$ with $t \neq \emptyset$, we have that the k -subsequence $(x'_s)_{s \in [L]^k}$ is plegma disjointly supported and admits $(y'_t)_{t \in [L]^{\leq k}}$ as a canonical tree decomposition.
- (v) For every $j \in \{1, \dots, k\}$ and $(s_i)_{i=1}^n \in Plm_n([L]^k)$, if I is the interval of \mathbb{N} with $\min I = \min \text{supp}(y_{s_1|j})$ and $\max I = \max \text{supp}(y_{s_n|j})$, then for every $1 \leq i \leq n$, $I(x_{s_i} - y_0) = y_{s_i|j}$.

The following is the main result of this section.

Theorem 33. *Let X be a Banach space with Schauder basis, $k \in \mathbb{N}$, $(x_s)_{s \in [\mathbb{N}]^k}$ be a k -sequence in X and $(\varepsilon_n)_n$ be a null sequence of positive reals. Assume that for some $M \in [\mathbb{N}]^\infty$, $(x_s)_{s \in [M]^k}$ is subordinated with respect to the weak topology of X and let x_0 be the weak limit of $(x_s)_{s \in [M]^k}$. Then there exist $L \in [M]^\infty$ and a k -subsequence $(\tilde{x}_s)_{s \in [L]^k}$ in X satisfying the following.*

- (i) $(\tilde{x}_s)_{s \in [L]^k}$ admits a canonical tree decomposition $(y_t)_{t \in [L]^{\leq k}}$ with $y_0 = x_0$.
- (ii) For every $s \in [L]^k$, $\|x_s - \tilde{x}_s\| < \varepsilon_n$, where $\min s = L(n)$.
- (iii) $(\tilde{x}_s)_{s \in [L]^k}$ is subordinated with respect to the weak topology of X . Moreover x_0 is the weak limit of $(\tilde{x}_s)_{s \in [L]^k}$.

Proof. Without loss of generality, we may assume that $(\varepsilon_n)_n$ is decreasing. We will first define a family $(y_t)_{t \in [M]^{\leq k}}$ of finitely supported vectors in X as follows. Let $\widehat{\varphi} : [M]^{\leq k} \rightarrow (X, w)$ be the continuous map witnessing that $(x_s)_{s \in [M]^k}$ is subordinated. For $t = \emptyset$, we set $y_0 = \widehat{\varphi}(\emptyset) = x_0$. For $t \in [M]^{\leq k} \setminus \{\emptyset\}$, let $w_t = \widehat{\varphi}(t) - \widehat{\varphi}(t \setminus \{\max t\})$. Notice that the sequence $(w_{t \cup \{m\}})_{m \in M}$ is weakly null, for all $t \in [M]^{\leq k}$. Hence, by a sliding hump argument, we may choose a family $\{I_t : t \in [M]^{\leq k} \setminus \{\emptyset\}\}$ of finite intervals of \mathbb{N} satisfying the following properties.

- (P1) For every $t \in [M]^{\leq k}$, with $t \neq \emptyset$, we have that $\|w_t - y_t\| = \|I_t^c(w_t)\| < \varepsilon_n/k$, where $M(n) = \max t$.
- (P2) For every $t \in [M]^{\leq k}$, $\min I_{t \cup \{m\}} \xrightarrow{m \in M} \infty$.

Now for every $t \in [M]^{\leq k} \setminus \{\emptyset\}$, we set $y_t = I_t(w_t)$ and the definition of the family $(y_t)_{t \in [M]^{\leq k}}$ is completed. Also, for every $s \in [M]^k$, we set $\tilde{x}_s = \sum_{t \sqsubseteq s} y_t$.

We claim that there exists $L \in [M]^\infty$ such that $(y_t)_{t \in [L]^{\leq k}}$ is a canonical tree decomposition of $(\tilde{x}_s)_{s \in [L]^k}$. Indeed, using (P2) and Ramsey's theorem, there exists $M_1 \in [M]^\infty$ such that for every $s \in [M_1]^k$ and $1 \leq j_1 < j_2 \leq k$, $\text{supp}(y_{s|j_1}) < \text{supp}(y_{s|j_2})$. Using again (P2) and Theorem 5, we find $M_2 \in [M_1]^\infty$ such that for every $(s_1, s_2) \in Plm_2([M_2]^k)$ and $1 \leq j_1 \leq j_2 \leq k$, $\text{supp}(y_{s_1|j_1}) < \text{supp}(y_{s_2|j_2})$, while for every $1 \leq j_1 < j_2 \leq k$, $\text{supp}(y_{s_2|j_1}) < \text{supp}(y_{s_1|j_2})$. We set $L = M_2$. By the above, we have that all conditions (i)-(v) of Definition 31 are fulfilled and therefore $(y_t)_{t \in [L]^{\leq k}}$ is a canonical tree decomposition of $(\tilde{x}_s)_{s \in [L]^k}$ and the proof of the claim is complete.

Notice that $x_s - \tilde{x}_s = \sum_{j=1}^k (w_s|j - y_s|j)$, for all $s \in [L]^k$. Hence by (P1) and since $(\varepsilon_n)_n$ is decreasing, we get that $\|x_s - \tilde{x}_s\| \leq \varepsilon_n$, where $L(n) = \min s$. It remains to show that $(\tilde{x}_s)_{s \in [L]^k}$ is subordinated. To this end, let $\tilde{\varphi} : [L]^{\leq k} \rightarrow X$ defined by $\tilde{\varphi}(t) = \sum_{u \sqsubseteq t} y_u$, for all $t \in [L]^{\leq k}$. Clearly $\tilde{\varphi}(\emptyset) = y_0 = \widehat{\varphi}(\emptyset)$ and $\tilde{x}_s = \tilde{\varphi}(s)$, for all $s \in [L]^k$. To show that $\tilde{\varphi}$ is continuous let $(t_n)_n$ be a sequence in $[L]^{\leq k}$ and $t \in [L]^{\leq k}$ such that $(t_n)_n$ converges to t . Setting $\max t_n = M(k_n)$, we may assume

that $k_n \rightarrow \infty$. Then

$$\|(\widehat{\varphi}(t_n) - \widehat{\varphi}(t)) - (\widetilde{\varphi}(t_n) - \widetilde{\varphi}(t))\| \leq \sum_{t \sqsubseteq u \sqsubseteq t_n} \|w_u - y_u\| \leq \varepsilon_{k_n} \xrightarrow{n \rightarrow \infty} 0$$

Since $\widehat{\varphi}(t_n) \xrightarrow{w} \widehat{\varphi}(t)$, we get that $\widetilde{\varphi}(t_n) \xrightarrow{w} \widetilde{\varphi}(t)$ and the proof is completed. \square

Notation 2. Let X be a Banach space and $k \in \mathbb{N}$. By $\mathcal{SM}_k^{wrc}(X)$ we will denote the set of all spreading sequences $(e_n)_n$ such that there exists a weakly relatively compact k -sequence of X which generates $(e_n)_n$ as a k -spreading model. Notice that $\mathcal{SM}_k^{wrc}(X) = \mathcal{SM}_k(X)$, for every reflexive space X and $k \in \mathbb{N}$.

Corollary 34. *Let X be a Banach space with Schauder basis and $k \in \mathbb{N}$. Then every $(e_n)_n \in \mathcal{SM}_k^{wrc}(X)$ is generated by a k -sequence in X which is subordinated with respect to the weak topology and admits a canonical tree decomposition.*

Proof. Let $k \in \mathbb{N}$ and $(x_s)_{s \in [\mathbb{N}]^k}$ be a weakly relatively compact k -sequence in X which generates a k -spreading model $(e_n)_n$. By Proposition 29, there exists $M \in [\mathbb{N}]^\infty$ such that $(x_s)_{s \in [M]^k}$ is subordinated. By Theorem 33, there exists $L \in [M]^\infty$ and a subordinated sequence $(\widetilde{x}_s)_{s \in [L]^k}$ in X which admits a canonical tree decomposition such that $\|x_s - \widetilde{x}_s\| < 1/n$, for every $s \in [L]^k$ with $\min s = L(n)$. Hence there is $N \in [L]^\infty$ such that $(\widetilde{x}_s)_{s \in [N]^k}$ also generates $(e_n)_n$ as a k -spreading model. Setting $z_s = \widetilde{x}_{N(s)}$, for all $s \in [\mathbb{N}]^k$, we have that $(z_s)_{s \in [\mathbb{N}]^k}$ is as desired. \square

6. NORM PROPERTIES OF SPREADING MODELS

In this section we provide conditions for k -sequences to admit unconditional, singular or trivial spreading models. Our main interest concerns subordinated k -sequences with respect to the weak topology.

6.1. Unconditional spreading models. As is well known every spreading model generated by a seminormalized weakly null sequence is an 1-unconditional spreading sequence. In this subsection we give an extension of this result for subordinated seminormalized weakly null k -sequences.

Lemma 35. *Let $k \in \mathbb{N}$ and $(x_s)_{s \in [\mathbb{N}]^k}$ be a k -sequence in a Banach space X . Suppose that $(x_s)_{s \in [\mathbb{N}]^k}$ is subordinated and let $\widehat{\varphi} : [\mathbb{N}]^{\leq k} \rightarrow (X, w)$ be the continuous map witnessing this. Let $\varepsilon > 0$, $M \in [\mathbb{N}]^\infty$ and $n \in \mathbb{N}$. Then for every $p \in \{1, \dots, n\}$ there exists a finite subset G of $[M]^k$ such that the following are satisfied.*

- (i) *There exists a convex combination $x = \sum_{s \in G} \mu_s x_s$ of $(x_s)_{s \in G}$ such that $\|\widehat{\varphi}(\emptyset) - x\| < \varepsilon$.*
- (ii) *For every $1 \leq i \leq n$ with $i \neq p$, there exists $s_i \in [M]^k$ such that for every $s_p \in G$, the family $(s_i)_{i=1}^n$ is a plegma family in $[M]^k$.*

Proof. For $k = 1$, the result follows by Mazur's theorem. We proceed by induction on $k \in \mathbb{N}$. Assume that the lemma is true for some $k \in \mathbb{N}$. We fix a subordinated $(k+1)$ -sequence $(x_s)_{s \in [\mathbb{N}]^{k+1}}$ in X , $M \in [\mathbb{N}]^\infty$, $n \in \mathbb{N}$, $\varepsilon > 0$ and $p \in \{1, \dots, n\}$.

Let $(x_t)_{t \in [M]^k}$ defined by $x_t = \widehat{\varphi}(t)$, for all $t \in [M]^k$. By our inductive assumption, there exists a finite subset F of $[M]^k$ satisfying the following.

- (a) *There exists a convex combination $\sum_{t \in F} \mu_t x_t$ of $(x_t)_{t \in F}$ such that*

$$(6) \quad \left\| \widehat{\varphi}(\emptyset) - \sum_{t \in F} \mu_t x_t \right\| < \varepsilon/2$$

- (b) For every $1 \leq i \leq n$ with $i \neq p$, there exists $t_i \in [M]^k$ such that for every $t_p \in F$, $(t_i)_{i=1}^n$ is a plegma family in $[M]^k$.

For notational simplicity we assume that $1 < p < n$ (the proof for $p \in \{1, n\}$ is similar). Pick $m_1 < \dots < m_{p-1}$ in M with $t_n(k) < m_1$ and set $s_i = t_i \cup \{m_i\}$, for all $i = 1, \dots, p-1$. Also let $M' = \{m \in M : m > m_{p-1}\}$. Since $\widehat{\varphi}$ is continuous, we have that $(x_{t \cup \{m\}})_{m \in M'} \xrightarrow{w} x_t$, for every $t \in F$. Hence by Mazur's theorem, for every $t \in F$, there exists a finite subset G_t of M' such that

$$(7) \quad \left\| x_t - \sum_{m \in G_t} \mu_m^t x_{t \cup \{m\}} \right\| < \varepsilon/2$$

for some convex combination $\sum_{m \in G_t} \mu_m^t x_{t \cup \{m\}}$ of $(x_{t \cup \{m\}})_{m \in G_t}$. We set

$$G = \{t \cup \{m\} : t \in F \text{ and } m \in G_t\}$$

Finally, pick $m_{p+1} < \dots < m_n$ in M with $\max\{m : m \in \bigcup_{t \in F} G_t\} < m_{p+1}$ and let $s_i = t_i \cup \{m_i\}$, for all $i = p+1, \dots, n$.

It is easy to check that every $(s_i)_{i=1}^n$ with $s_p \in G$, is a plegma family in $[M]^{k+1}$. It remains to show that condition (i) of the lemma is also satisfied. To this end, let $\mu_s = \mu_t \mu_m^t$, for every $s = t \cup \{m\} \in G$, where $\max t < m$. Notice that

$$\sum_{s \in G} \mu_s = \sum_{t \in F} \mu_t \sum_{m \in G_t} \mu_m^t = \sum_{t \in F} \mu_t = 1$$

and therefore $\sum_{s \in G} \mu_s x_s$ is a convex combination of $(x_s)_{s \in G}$. Moreover, we have

$$\begin{aligned} \left\| \widehat{\varphi}(\emptyset) - \sum_{s \in G} \mu_s x_s \right\| &= \left\| \widehat{\varphi}(\emptyset) - \sum_{t \in F} \mu_t \sum_{m \in G_t} \mu_m^t x_{t \cup \{m\}} \right\| \\ &\leq \left\| \widehat{\varphi}(\emptyset) - \sum_{t \in G'} \mu_t' x_t \right\| + \sum_{t \in F} \mu_t \cdot \left\| x_t - \sum_{m \in G_t} \mu_m^t x_{t \cup \{m\}} \right\| \stackrel{(6),(7)}{<} \varepsilon \end{aligned}$$

and the proof is complete. \square

Theorem 36. *Let $k \in \mathbb{N}$ and $(x_s)_{s \in [\mathbb{N}]^k}$ be a k -sequence in a Banach space X . Suppose that $(x_s)_{s \in [\mathbb{N}]^k}$ is seminormalized, subordinated (with respect to the weak topology of X) and weakly null. Then every k -spreading model of $(x_s)_{s \in [\mathbb{N}]^k}$ is 1-unconditional.*

Proof. Let $(e_n)_n$ be a spreading model of $(x_s)_{s \in [\mathbb{N}]^k}$. Lemma 35 and the averaging technique used for the proof of the corresponding result in the case of the classical spreading models (see [5] Proposition I.5.1) yield that for every $n \in \mathbb{N}$, $1 \leq p \leq n$, $a_1, \dots, a_n \in [-1, 1]$ and $\varepsilon > 0$, we have

$$\left\| \sum_{\substack{i=1 \\ i \neq p}}^n a_i e_i \right\|_* \leq \left\| \sum_{i=1}^n a_i e_i \right\|_* + \varepsilon$$

Since the above inequality holds for every $\varepsilon > 0$, we have that

$$(8) \quad \left\| \sum_{\substack{i=1 \\ i \neq p}}^n a_i e_i \right\|_* \leq \left\| \sum_{i=1}^n a_i e_i \right\|_*$$

for all $n \in \mathbb{N}$, $1 \leq p \leq n$ and $a_1, \dots, a_n \in [-1, 1]$. Since $(x_s)_{s \in [\mathbb{N}]^k}$ is seminormalized, we have that $\|e_1\|_* > 0$. By (8) we get that $\|e_1 - e_2\|_* > 0$. By Proposition 13, we get that $(e_n)_n$ is non trivial. An iterated use of (8) completes the proof. \square

We close this subsection by giving an example showing that for $k \geq 2$ the assumption in Theorem 36 that the k -sequence is subordinated is necessary. More precisely, for every $k \geq 2$, there exist seminormalized weakly null k -sequences which generate conditional Schauder basic spreading models.

Example 2. For simplicity we state the example for $k = 2$. Let $(e_n)_n$ be the usual basis of c_0 and $(x_s)_{s \in [\mathbb{N}]^2}$ be the 2-sequence in c_0 , defined by $x_s = \sum_{n=\min s}^{\max s} e_n$, for all $s \in [\mathbb{N}]^2$. Clearly, $(x_s)_{s \in [\mathbb{N}]^2}$ is a normalized weakly null 2-sequence. It is easy to check that for all $l \in \mathbb{N}$, $a_1, \dots, a_l \in \mathbb{R}$ and $(s_j)_{j=1}^l \in Plm_l([\mathbb{N}]^2)$, we have

$$\left\| \sum_{j=1}^l a_j x_{s_j} \right\| = \max \left(\max_{1 \leq k \leq l} \left| \sum_{j=1}^k a_j \right|, \max_{1 \leq k \leq l} \left| \sum_{j=k}^l a_j \right| \right)$$

Therefore every spreading model of $(x_s)_{s \in [\mathbb{N}]^2}$, is equivalent to the summing basis.

6.2. Singular and trivial spreading models. The results of this subsection concern the k -spreading models generated by subordinated k -sequences which are not weakly null.

Lemma 37. *Let X be a Banach space, $k \in \mathbb{N}$, $(x_s)_{s \in [\mathbb{N}]^k}$ be a k -sequence in X and $x_0 \in X$. Let $x'_s = x_s - x_0$, for all $s \in [\mathbb{N}]^k$ and assume that $(x_s)_{s \in [\mathbb{N}]^k}$ and $(x'_s)_{s \in [\mathbb{N}]^k}$ generate k -spreading models $(e_n)_n$ and $(\tilde{e}_n)_n$ respectively. Then the following hold.*

- (a) $\left\| \sum_{i=1}^n a_i e_i \right\| = \left\| \sum_{i=1}^n a_i \tilde{e}_i \right\|$, for every $n \in \mathbb{N}$ and $a_1, \dots, a_n \in \mathbb{R}$ with $\sum_{i=1}^n a_i = 0$.
- (b) The sequence $(e_n)_n$ is trivial if and only if $(\tilde{e}_n)_n$ is trivial.
- (c) The sequence $(e_n)_n$ is equivalent to the usual basis of ℓ^1 if and only if $(\tilde{e}_n)_n$ is equivalent to the usual basis of ℓ^1 .

Proof. (a) Notice that for every $n \in \mathbb{N}$, s_1, \dots, s_n in $[\mathbb{N}]^k$ and $a_1, \dots, a_n \in \mathbb{R}$ with $\sum_{i=1}^n a_i = 0$, we have $\sum_{i=1}^n a_i x_{s_i} = \sum_{i=1}^n a_i x'_{s_i}$. Since $(e_n)_n$ and $(\tilde{e}_n)_n$ are generated by $(x_s)_{s \in [\mathbb{N}]^k}$ and $(x'_s)_{s \in [\mathbb{N}]^k}$ the result follows.

(b) It follows by assertion (a) and Proposition 13.

(c) We fix $\varepsilon > 0$. If $(\tilde{e}_n)_n$ is not equivalent to the usual basis of ℓ^1 then there exist $n \in \mathbb{N}$ and $a'_1, \dots, a'_n \in \mathbb{R}$ such that $\sum_{i=1}^n |a'_i| = 1$ and $\left\| \sum_{i=1}^n a'_i \tilde{e}_i \right\| < \varepsilon$. Setting $a_i = a'_i/2$ and $a_{n+i} = -a'_i/2$, for all $1 \leq i \leq n$, we have $\sum_{i=1}^{2n} a_i = 0$ and therefore, $\left\| \sum_{i=1}^{2n} a_i e_i \right\| = \left\| \sum_{i=1}^{2n} a_i \tilde{e}_i \right\| < \varepsilon$. Since $\sum_{i=1}^{2n} |a_i| = 1$, $(e_n)_n$ is also not equivalent to the usual basis of ℓ^1 . \square

Theorem 38. *Let X be a Banach space, $k \in \mathbb{N}$ and $(x_s)_{s \in [\mathbb{N}]^k}$ be a subordinated k -sequence in X . Also let $x'_s = x_s - x_0$, for every $s \in [\mathbb{N}]^k$, where x_0 is the weak limit of $(x_s)_{s \in [\mathbb{N}]^k}$. Assume that for some $M \in [\mathbb{N}]^\infty$ the k -subsequence $(x_s)_{s \in [M]^k}$ generates a non trivial k -spreading model $(e_n)_n$. If $x_0 \neq 0$, then exactly one of the following holds.*

- (i) The sequence $(e_n)_n$ as well as every spreading model of $(x'_s)_{s \in [M]^k}$ is equivalent to the usual basis of ℓ^1 .
- (ii) The sequence $(e_n)_n$ is singular and if $e_n = e'_n + e$ is its natural decomposition then $(e'_n)_n$ is the unique k -spreading model of $(x'_s)_{s \in [M]^k}$ and $\|e\| = \|x_0\|$.

Proof. Let $(\tilde{e}_n)_n$ be a k -spreading model of $(x'_s)_{s \in [M]^k}$. If $(e_n)_n$ is equivalent to the usual basis of ℓ^1 then by Lemma 37, we have that the same holds for $(\tilde{e}_n)_n$ and hence (i) is satisfied.

Assume for the following that $(e_n)_n$ is not equivalent to the usual basis of ℓ^1 . Since it is also non trivial, by Lemma 37, we have that $(\tilde{e}_n)_n$ is non trivial and not equivalent to the ℓ^1 -basis. Let $L \in [M]^\infty$ such that $(x'_s)_{s \in [L]^k}$ generates $(\tilde{e}_n)_n$. Since $(\tilde{e}_n)_n$ is non trivial, it is easy to see that $(x'_s)_{s \in [L]^k}$ is seminormalized. Also notice that $(x'_s)_{s \in [M]^k}$ is subordinated and weakly null. Therefore by Theorem 36, $(\tilde{e}_n)_n$ is 1-unconditional. Moreover, since $(\tilde{e}_n)_n$ is not equivalent to the usual basis of ℓ^1 , by Proposition 14, we conclude that $(\tilde{e}_n)_n$ is Cesàro summable to zero. Hence we have

$$(9) \quad \lim_{n \rightarrow \infty} \left\| \frac{1}{n} \sum_{j=1}^n e_j - \frac{1}{n} \sum_{j=n+1}^{2n} e_j \right\| = \lim_{n \rightarrow \infty} \left\| \frac{1}{n} \sum_{j=1}^n \tilde{e}_j - \frac{1}{n} \sum_{j=n+1}^{2n} \tilde{e}_j \right\| = 0$$

Also it is easy to see that

$$(10) \quad \left\| \frac{1}{n} \sum_{j=1}^n e_j \right\| \rightarrow \|x_0\| > 0$$

By (9) and (10), we get that $(e_n)_n$ is not Schauder basic, i.e. it is singular. Let $e_n = e'_n + e$ be the natural decomposition of $(e_n)_n$. By (10) and the fact that $(e'_n)_n$ is Cesàro summable to zero, we have that $\|e\| = \|x_0\|$. To complete the proof it remains to show that $(\tilde{e}_n)_n$ and $(e'_n)_n$ are isometrically equivalent. Indeed, we fix $n \in \mathbb{N}$ and $a_1, \dots, a_n \in \mathbb{R}$. For every $p \in \mathbb{N}$, let $(s_j^p)_{j=1}^{n+p} \in Plm_p([L]^k)$ such that $s_1^p(1) \geq L(n+p)$. We also set $a = \sum_{j=1}^n a_j$. Then we have

$$\begin{aligned} \left\| \sum_{j=1}^n a_j e'_j \right\| &= \lim_{p \rightarrow \infty} \left\| \sum_{j=1}^n a_j e'_j - \frac{a}{p} \sum_{j=n+1}^{n+p} e'_j \right\| = \lim_{p \rightarrow \infty} \left\| \sum_{j=1}^n a_j e_j - \frac{a}{p} \sum_{j=n+1}^{n+p} e_j \right\| \\ &= \lim_{p \rightarrow \infty} \left\| \sum_{j=1}^n a_j x_{s_j^p} - \frac{a}{p} \sum_{j=n+1}^{n+p} x_{s_j^p} \right\| = \lim_{p \rightarrow \infty} \left\| \sum_{j=1}^n a_j x'_{s_j^p} - \frac{a}{p} \sum_{j=n+1}^{n+p} x'_{s_j^p} \right\| \\ &= \lim_{p \rightarrow \infty} \left\| \sum_{j=1}^n a_j \tilde{e}_j - \frac{a}{p} \sum_{j=n+1}^{n+p} \tilde{e}_j \right\| = \left\| \sum_{j=1}^n a_j \tilde{e}_j \right\| \end{aligned}$$

□

By Remark 5, Proposition 29 and Theorems 36 and 38, we derive the following.

Corollary 39. *Let X be a Banach space, $k \in \mathbb{N}$ and $(e_n)_n \in \mathcal{SM}_k^{wrc}(X)$ non trivial. Then one of the following holds.*

- (i) *The sequence $(e_n)_n$ is unconditional.*
- (ii) *The sequence $(e_n)_n$ is singular and if $e_n = e'_n + e$ is the natural decomposition of $(e_n)_n$ then $(e'_n)_n \in \mathcal{SM}_k^{wrc}(X)$, $(e'_n)_n$ is unconditional, weakly null and Cesàro summable to zero. Moreover, the spaces generated by $(e_n)_n$ and $(e'_n)_n$ are isomorphic.*

The next theorem provides more information concerning the trivial k -spreading models. Since we shall not use this result in the sequel, we omit its proof.

Theorem 40. *Let $k \in \mathbb{N}$, $(x_s)_{s \in [\mathbb{N}]^k}$ be a k -sequence in a Banach space X and $(E, \|\cdot\|_*)$ be an infinite dimensional seminormed linear space with Hamel basis $(e_n)_n$. Assume that for some $M \in [\mathbb{N}]^\infty$, the k -subsequence $(x_s)_{s \in [M]^k}$ generates $(e_n)_n$ as an k -spreading model. Then the following are equivalent:*

- (i) *The sequence $(e_n)_n$ is trivial.*
- (ii) *The seminorm $\|\cdot\|_*$ is not a norm on E .*
- (iii) *$(x_s)_{s \in [M]^k}$ contains a further norm Cauchy k -subsequence, i.e. there exists $L \in [M]^\infty$ such that for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ satisfying that $\|x_s - x_t\| < \varepsilon$, for all $s, t \in [L]^k$ with $n_0 \leq \min\{\min s, \min t\}$.*
- (iv) *There exists $x \in X$ such that every k -subsequence of $(x_s)_{s \in [M]^k}$ contains a further k -subsequence convergent to x .*

7. COMPOSITION OF THE SPREADING MODELS

In this section we study the composition property of the k -spreading models. Moreover we recall the definition of the k -iterated spreading models and we investigate their relation with the k -spreading models. We start with the following definition.

Definition 41. *Let X be a Banach space with a Schauder basis and $k \in \mathbb{N}$. Then a k -spreading model $(e_n)_n$ of X will be called *plegma block generated* if there exists a k -sequence $(x_s)_{s \in [\mathbb{N}]^k}$ which is plegma block and generates $(e_n)_n$ as a k -spreading model.*

Remark 8. By Lemma 22, we easily conclude that for $1 \leq k_1 < k_2$, every plegma block generated k_1 -spreading model is also a plegma block k_2 -spreading model. Thus the plegma block k -spreading models of a Banach space X with a Schauder basis form an increasing hierarchy.

Theorem 42. *Let X be a Banach space, $k \in \mathbb{N}$ and $(e_n)_n \in \mathcal{SM}_k(X)$ such that $(e_n)_n$ is a Schauder basic sequence. Let E be the Banach space with Schauder basis the sequence $(e_n)_n$, $d \in \mathbb{N}$ and $(\tilde{e}_n)_n$ be a plegma block generated d -spreading model of E . Then $(\tilde{e}_n)_n \in \mathcal{SM}_{k+d}(X)$.*

Proof. We fix a plegma block d -sequence $(y_t)_{t \in [\mathbb{N}]^d}$ in E which generates $(\tilde{e}_n)_n$ as a d -spreading model with respect to some null sequence $(\tilde{\delta}_n)_n$ of positive reals. By Proposition 19, we may also choose a k -sequence $(x_s)_{s \in [\mathbb{N}]^k}$ in X which generates $(e_n)_n$ as a k -spreading model with respect to the same sequence $(\tilde{\delta}_n)_n$.

Since $(y_t)_{t \in [\mathbb{N}]^d}$ is finitely supported, setting for every $t \in [\mathbb{N}]^d$, $F_t = \text{supp}(y_t)$,

$$(11) \quad y_t = \sum_{j=1}^{|F_t|} a_{F_t(j)}^t e_{F_t(j)}$$

For every $v \in [\mathbb{N}]^{k+d}$, let t_v (resp. s_v) be the unique element in $[\mathbb{N}]^d$ (resp. $[\mathbb{N}]^k$) such that $v = t_v \cup s_v$ and $t_v < s_v$. For every $v \in [\mathbb{N}]^{k+d}$ and $j \in \{1, \dots, |F_{t_v}|\}$, we set

$$(12) \quad s_j^v = (s_v(1) + j - 1, \dots, s_v(k) + j - 1)$$

Notice that $(s_j^v)_{j=1}^{|F_{t_v}|}$ is a finite sequence in $[\mathbb{N}]^k$ with $s_1^v = s_v$.

We define a $(k+d)$ -sequence $(z_v)_{v \in [\mathbb{N}]^{k+d}}$ in X , by setting

$$(13) \quad z_v = \sum_{j=1}^{|F_{t_v}|} a_{F_{t_v}(j)}^{t_v} x_{s_j^v}$$

The proof will be completed once we show the following.

Claim 1. There exists $M \in [\mathbb{N}]^\infty$ such that $(z_v)_{v \in [M]^{k+d}}$ generates $(\tilde{e}_n)_n$ as a $(k+d)$ -spreading model.

Proof of Claim 1: For every $l \in \mathbb{N}$, we define a family $\mathcal{A}_l \subseteq Plm_l([\mathbb{N}]^{k+d})$ as follows:

$$\mathcal{A}_l = \left\{ (v_i)_{i=1}^l \in Plm_l([\mathbb{N}]^{k+d}) : s_1^{v_1}(1) \geq \sum_{i=1}^l |F_{t_{v_i}}| \right. \\ \left. \text{and } (s_j^{v_1})_{j=1}^{|F_{t_{v_1}}|} \cap \dots \cap (s_j^{v_l})_{j=1}^{|F_{t_{v_l}}|} \in Plm_{\sum_{i=1}^l |F_{t_{v_i}}|}([\mathbb{N}]^k) \right\}$$

Using (12), the fact that for every $(v_i)_{i=1}^l \in Plm_l([\mathbb{N}]^{k+d})$, $(s_{v_i})_{i=1}^l \in Plm_l([\mathbb{N}]^k)$ and that $s_1^{v_i} = s_{v_i}$, for all $1 \leq i \leq l$, it is easy to check that $\mathcal{A}_l \cap Plm_l([L]^{k+d}) \neq \emptyset$, for every $l \in \mathbb{N}$ and $L \in [\mathbb{N}]^\infty$. Hence, an iterated use of Theorem 5, yields an $L \in [\mathbb{N}]^\infty$ such that $(v_i)_{i=1}^l \in \mathcal{A}_l$, for every $(v_i)_{i=1}^l \in Plm_l([L]^{k+d})$, with $v_1(1) \geq L(l)$.

We fix $l \in \mathbb{N}$, $(v_i)_{i=1}^l \in Plm_l([L]^{k+d})$ with $v_1(1) \geq L(l)$ and $a_1, \dots, a_l \in [-1, 1]$. Notice that

$$(14) \quad \left\| \sum_{i=1}^l a_i z_{v_i} \right\| - \left\| \sum_{i=1}^l a_i \tilde{e}_i \right\| \leq \left\| \sum_{i=1}^l a_i z_{v_i} \right\| - \left\| \sum_{i=1}^l a_i y_{t_{v_i}} \right\| \\ + \left\| \sum_{i=1}^l a_i y_{t_{v_i}} \right\| - \left\| \sum_{i=1}^l a_i \tilde{e}_i \right\|$$

Also observe that $(t_{v_i})_{i=1}^l \in Plm_l([L]^d)$ and $t_{v_1}(1) = v_1(1) \geq L(l) \geq l$. Hence,

$$(15) \quad \left\| \sum_{i=1}^l a_i y_{t_{v_i}} \right\| - \left\| \sum_{i=1}^l a_i \tilde{e}_i \right\| < \tilde{\delta}_l$$

Also, $s_1^{v_1}(1) \geq \sum_{i=1}^l |F_{t_{v_i}}|$ and $F_{t_{v_1}} < \dots < F_{t_{v_l}}$. Therefore,

$$(16) \quad \left\| \sum_{i=1}^l a_i z_{v_i} \right\| - \left\| \sum_{i=1}^l a_i y_{t_{v_i}} \right\| = \left\| \sum_{i=1}^l \sum_{j=1}^{l_{t_{v_i}}} a_i a_{F_{t_{v_i}}(j)}^{t_{v_i}} x_{s_j^{v_i}} \right\| \\ - \left\| \sum_{i=1}^l \sum_{j=1}^{l_{t_{v_i}}} a_i a_{F_{t_{v_i}}(j)}^{t_{v_i}} e_{F_{t_{v_i}}(j)} \right\| < 2CK\tilde{\delta}_l$$

where C is the basis constant of $(e_n)_n$ and $K = \sup\{\|y_t\| : t \in [L]^k\}$.

By (14), (15) and (16), we obtain that for every $l \in \mathbb{N}$, $(v_i)_{i=1}^l \in Plm_l([L]^k)$ with $v_1(1) \geq L(l)$ and $a_1, \dots, a_l \in [-1, 1]$, we have

$$\left\| \sum_{i=1}^l a_i z_{v_i} \right\| - \left\| \sum_{i=1}^l a_i \tilde{e}_i \right\| < \delta_l$$

where $\delta_l = (1 + 2CK)\tilde{\delta}_l$. By Lemma 20, there exists $M \in [L]^\infty$, such that $(z_v)_{v \in [M]^k}$ generates $(\tilde{e}_n)_n$ as a k -spreading model and the proof of the claim as well as of Theorem 42 is complete. \square

Corollary 43. *Let X be a Banach space and Y be either ℓ^p for some $p \in [1, \infty)$ or c_0 . Also let $k \in \mathbb{N}$, $(e_n)_n \in \mathcal{SM}_k^{wrc}(X)$ be non trivial and E be the Banach space generated by $(e_n)_n$. Suppose that E contains an isomorphic copy of Y . Then $\mathcal{SM}_{k+1}(X)$ contains a sequence equivalent to the usual basis of Y .*

Proof. First assume that $(e_n)_n$ is Schauder basic. Notice that E contains a block sequence $(y_n)_n$ equivalent to the usual basis of Y . It is easy to see that $(y_n)_n$ admits a spreading model $(\tilde{e}_n)_n$ equivalent to the usual basis of Y . By Theorem 42 we have that $(\tilde{e}_n)_n \in \mathcal{SM}_{k+1}(X)$.

Assume now that $(e_n)_n$ is not Schauder basic. Since $(e_n)_n$ is non trivial, we have that $(e_n)_n$ is singular. Let $e_n = e'_n + e$ be its natural decomposition and E' the space generated by $(e'_n)_n$. By Remark 5 we have that E and E' are isomorphic and therefore E' contains an isomorphic copy of Y . By Corollary 39 we have that $(e'_n)_n \in \mathcal{SM}_{k+1}(X)$. Since $(e'_n)_n$ is unconditional, the result follows as in the first case. \square

7.1. The k -iterated spreading models. In this subsection we define the k -iterated spreading models of a Banach space X which although they have not been named, have been appeared in [6] and [17]. We also study their relation with the k -spreading models.

Definition 44. *The k -iterated spreading models of a Banach space X are inductively defined as follows. The 1-iterated are the non trivial spreading models of X . Assume that for some $k \in \mathbb{N}$ the k -iterated spreading models of X have been defined. Then the $(k + 1)$ -iterated spreading models are the non trivial spreading models of the spaces generated by the k -iterated spreading models.*

Notice that the class of the k -iterated spading models of a Banach space X is contained in the one of the $(k + 1)$ -iterated spreading models. In the sequel we provide a sufficient condition ensuring that the k -iterated spreading models of a Banach space X are up to isomorphism contained in $\mathcal{SM}_k(X)$. To this end we need the following lemma.

Lemma 45. *Let X be a Banach space and $k \in \mathbb{N}$. Let $(e_n^0)_n$ be a Schauder basic k -spreading model of X , E_0 be the space generated by $(e_n^0)_n$, $(e_n)_n$ be a non trivial spreading model of E_0 and E be the space generated by $(e_n)_n$. If E_0 is reflexive then there exists an unconditional $(k + 1)$ -spreading model of X generating a space isomorphic to E .*

Proof. Let $(x_n)_n$ be a sequence in E_0 generating $(e_n)_n$ as a spreading model. Since E_0 is reflexive, we may assume that $(x_n)_n$ is weakly convergent to some $x_0 \in E_0$. If $x_0 = 0$, then $(e_n)_n$ is unconditional and it is generated by a block sequence in E_0 , while if $(e_n)_n$ is equivalent to the usual basis of ℓ^1 then E_0 contains a block sequence generating an ℓ^1 spreading model. Therefore, in both cases the result follows by Theorem 42. Assume that $x_0 \neq 0$ and $(e_n)_n$ is not equivalent to the usual basis of ℓ^1 . Let $x'_n = x_n - x_0$, for all $n \in \mathbb{N}$. By Theorem 38, we have that $(e_n)_n$ is singular and $(e'_n)_n$ is the unique spreading model of $(x'_n)_n$, where $e_n = e'_n + e$ is the natural decomposition of $(e_n)_n$. Since $(x'_n)_n$ is weakly null, we have that $(e'_n)_n$

is generated by a block sequence in E_0 as a spreading model. Hence, by Theorem 42, the sequence $(e'_n)_n$ is a $(k+1)$ -spreading model of X . Moreover, by Remark 5, $(e'_n)_n$ is unconditional and the space E' generated by $(e'_n)_n$ is isomorphic to E . \square

Proposition 46. *Let X be a reflexive space and $k \in \mathbb{N}$ such that every space generated by a k -iterated spreading model of X is reflexive. Then every space generated by a $(k+1)$ -iterated spreading model of X is isomorphic to the space generated by an unconditional $(k+1)$ -spreading model of X .*

Proof. We first treat the case $k = 1$. So assume that X as well as every space generated by a spreading model of X is reflexive. Let $(\tilde{e}_n)_n$ be a 2-iterated spreading model of X and \tilde{E} be the space generated by $(\tilde{e}_n)_n$. Also let \tilde{E}_0 be the space generated by a spreading model of X such that $(\tilde{e}_n)_n$ is a spreading model of \tilde{E}_0 . Since X is reflexive, by Corollary 39, we conclude that \tilde{E}_0 is isomorphic to a space E_0 , generated by an unconditional spreading model of X . Moreover, by our assumption E_0 is also reflexive. Summarizing, the space E_0 is reflexive, it has a Schauder basis which is a spreading model of X and it is isomorphic to \tilde{E}_0 . Therefore, E_0 admits a spreading model $(e_n)_n$ equivalent to $(\tilde{e}_n)_n$. Let E be the space generated by $(e_n)_n$. By Lemma 45, there exists an unconditional 2-spreading model of X generating a space isomorphic to E . Since E is isomorphic to \tilde{E} the proof of the proposition for $k = 1$ is completed.

We proceed by induction. Assume that the proposition holds for some $k \in \mathbb{N}$ and let X be a reflexive space such that every space generated by a $(k+1)$ -iterated spreading model of X is reflexive. Let $(\tilde{e}_n)_n$ be a $(k+2)$ -iterated spreading model of X and \tilde{E} be the space that it generates. Let \tilde{E}_0 be the space generated by a $(k+1)$ -iterated spreading model of X admitting $(\tilde{e}_n)_n$ as a spreading model. Since the k -iterated spreading models of X are included in the $(k+1)$ -iterated ones, we have that the spaces generated by the k -iterated spreading models of X are reflexive. Hence, by our assumption that the proposition holds for the positive integer k , we have that \tilde{E}_0 is isomorphic to some space E_0 generated by an unconditional $(k+1)$ -spreading model of X . Therefore, E_0 is reflexive, it is generated by a Schauder basic $(k+1)$ -spreading model of X and admits a spreading model $(e_n)_n$ equivalent to $(\tilde{e}_n)_n$. Let E be the space generated by $(e_n)_n$. By Lemma 45, there exists an unconditional $k+2$ -spreading model of X generating a space isomorphic to E . Since E is isomorphic to \tilde{E} the proof of is completed. \square

Corollary 47. *Let X be a reflexive space such that for every $k \in \mathbb{N}$, every space generated by an unconditional k -spreading model of X is reflexive. Then for every $k \in \mathbb{N}$, every space generated by a k -iterated spreading model of X is isomorphic to the space generated by an unconditional k -spreading model of X .*

Proof. By Corollary 39 we have that every space generated by a spreading model of X is isomorphic to the space generated by an unconditional spreading model of X and therefore it is reflexive. The proof is carried out by induction and using Proposition 46. \square

Remark 9. As it is well known, see [5], every non trivial spreading model of c_0 generates a space isomorphic to c_0 . This easily implies that every k -iterated spreading model of c_0 generates a space isomorphic to c_0 . On the other hand, as we will see in Section 10, the class of the 2-spreading models of c_0 includes all

spreading bimonote Schauder basic sequences yielding the existence of 2-spreading models which are not 2-iterated ones.

Remark 10. H.P. Rosenthal had asked whether every 2-iterated spreading model of a Banach space X is actually a classical one. In [6] a Banach space X has been constructed not admitting ℓ^1 as a spreading model, while there is a spreading model generating a space which contains ℓ^1 . Thus ℓ^1 occurs as 2-iterated spreading model but not as a classical one. A more striking result (see [2]) asserts the existence of a Banach space X not admitting ℓ^1 as a spreading model but ℓ^1 is isomorphic to a subspace of every space generated by a non trivial spreading model of X . It remains open if for every $k \in \mathbb{N}$ there exists a Banach space X_{k+1} such that the class of $(k+1)$ -iterated spreading models strictly includes the corresponding one of k -iterated.

8. k -SPREADING MODELS EQUIVALENT TO THE ℓ^1 BASIS

In this section we study the properties of the k -spreading models equivalent to the usual basis of ℓ^1 .

8.1. Splitting spreading sequences equivalent to the ℓ^1 basis. In this subsection we present some stability properties of spreading sequences in seminormed linear spaces which are actually related to the non distortion of ℓ^1 (c.f. [11]).

Let $(e_n)_n$ be a spreading sequence in a seminormed linear space $(E, \|\cdot\|_*)$ and $c > 0$. We say that $(e_n)_n$ admits a lower ℓ^1 -estimate of constant c , if for every $n \in \mathbb{N}$ and $a_1, \dots, a_n \in \mathbb{R}$, we have $c \sum_{i=1}^n |a_i| \leq \left\| \sum_{i=1}^n a_i e_i \right\|_*$.

Proposition 48. Let $(E, \|\cdot\|_o), (E_1, \|\cdot\|_*)$, $(E_2, \|\cdot\|_{**})$ be seminormed linear spaces and $(e_n)_n, (e_n^1)_n$ and $(e_n^2)_n$ be spreading sequences in E, E_1 and E_2 respectively. Assume that for every $n \in \mathbb{N}$ and $a_1, \dots, a_n \in \mathbb{R}$, we have

$$(17) \quad \left\| \sum_{i=1}^n a_i e_i \right\|_o \leq \left\| \sum_{i=1}^n a_i e_i^1 \right\|_* + \left\| \sum_{i=1}^n a_i e_i^2 \right\|_{**}$$

If $(e_n)_n$ admits a lower ℓ^1 -estimate of constant $c > 0$ and $(e_n^2)_n$ does not admit any lower ℓ^1 -estimate then $(e_n^1)_n$ admits a lower ℓ^1 -estimate of the same constant c .

Proof. Suppose on the contrary that $(e_n^1)_n$ does not admit a lower ℓ^1 -estimate of constant c . Then there exist $\varepsilon > 0$, $n \in \mathbb{N}$ and $a_1, \dots, a_n \in \mathbb{R}$ with $\sum_{i=1}^n |a_i| = 1$ such that $\left\| \sum_{i=1}^n a_i e_i^1 \right\|_* < c - \varepsilon$. Also since $(e_n^2)_n$ does not admit any lower ℓ^1 -estimate, there exist $m \in \mathbb{N}$ and $b_1, \dots, b_m \in \mathbb{R}$ such that $\sum_{j=1}^m |b_j| = 1$ and $\left\| \sum_{j=1}^m b_j e_j^2 \right\|_{**} < \varepsilon/2$. Hence, we get that

$$(18) \quad \left\| \sum_{i=1}^n \sum_{j=1}^m a_i \cdot b_j e_{(i-1)m+j}^1 \right\|_* \leq \sum_{j=1}^m |b_j| \left\| \sum_{i=1}^n a_i e_{(i-1)m+j}^1 \right\|_* < c - \varepsilon$$

and similarly

$$(19) \quad \left\| \sum_{i=1}^n \sum_{j=1}^m a_i \cdot b_j e_{(i-1)m+j}^2 \right\|_{**} \leq \sum_{i=1}^n |a_i| \left\| \sum_{j=1}^m b_j e_{(i-1)m+j}^2 \right\|_{**} < \frac{\varepsilon}{2}$$

But then by (17), we obtain that

$$\begin{aligned} \left\| \sum_{i=1}^n \sum_{j=1}^m a_i \cdot b_j e_{(i-1)m+j} \right\|_{\circ} &\leq \left\| \sum_{i=1}^n \sum_{j=1}^m a_i \cdot b_j e_{(i-1)m+j}^1 \right\|_* \\ &+ \left\| \sum_{i=1}^n \sum_{j=1}^m a_i \cdot b_j e_{(i-1)m+j}^2 \right\|_{**} \stackrel{(18),(19)}{<} c - \frac{\varepsilon}{2} \end{aligned}$$

which since $\sum_{i=1}^n \sum_{j=1}^m |a_i| \cdot |b_j| = 1$, contradicts that $(e_n)_n$ admits a lower ℓ^1 -estimate of constant c . \square

Corollary 49. *Let $k \in \mathbb{N}$ and $(x_s)_{s \in [\mathbb{N}]^k}, (x_s^1)_{s \in [\mathbb{N}]^k}, (x_s^2)_{s \in [\mathbb{N}]^k}$ be three k -sequences in a Banach space X such that for all $s \in [\mathbb{N}]^k$, $x_s = x_s^1 + x_s^2$. Assume that the k -sequences $(x_s)_{s \in [\mathbb{N}]^k}$, $(x_s^1)_{s \in [\mathbb{N}]^k}$ and $(x_s^2)_{s \in [\mathbb{N}]^k}$ generate the sequences $(e_n)_n$, $(e_n^1)_n$ and $(e_n^2)_n$ respectively, as k -spreading models. If $(e_n)_n$ admits a lower ℓ^1 -estimate of constant $c > 0$ and $(e_n^2)_n$ does not admit any lower ℓ^1 -estimate then $(e_n^1)_n$ admits a lower ℓ^1 -estimate of constant c .*

Proof. For every $n \in \mathbb{N}$, $a_1, \dots, a_n \in \mathbb{R}$ and $(s_j)_{j=1}^n$ in $[\mathbb{N}]^k$, we have

$$(20) \quad \left\| \sum_{j=1}^n a_j x_{s_j} \right\| \leq \left\| \sum_{j=1}^n a_j x_{s_j}^1 \right\| + \left\| \sum_{j=1}^n a_j x_{s_j}^2 \right\|$$

Let $(E, \|\cdot\|_{\circ}), (E_1, \|\cdot\|_*)$, $(E_2, \|\cdot\|_{**})$ be the seminormed linear spaces with Hamel bases $(e_n)_n$, $(e_n^1)_n$ and $(e_n^2)_n$ respectively. Notice that (20) implies that (17) holds and therefore the conclusion follows by Proposition 48. \square

8.2. k -spreading models almost isometric to the ℓ^1 basis. Let $c > 0$, $k \in \mathbb{N}$ and $(x_s)_{s \in [\mathbb{N}]^k}$ be a k -sequence in a Banach space X . We will say that the k -sequence $(x_s)_{s \in [\mathbb{N}]^k}$ generates ℓ^1 as a k -spreading model of constant c , if $(x_s)_{s \in [\mathbb{N}]^k}$ generates a k -spreading model $(e_n)_n$ which admits a lower ℓ_1 -estimate of constant c .

Proposition 50. *Let X be a Banach space and $k \in \mathbb{N}$. Assume that X admits a k -spreading model equivalent to the usual basis of ℓ^1 . Then for every $\varepsilon > 0$ there exists a k -sequence $(y_s)_{s \in [\mathbb{N}]^k}$ in X with $1 - \varepsilon \leq \|y_s\| \leq 1$, for every $s \in [\mathbb{N}]^k$, which generates ℓ^1 as a k -spreading model of constant $1 - \varepsilon$.*

Proof. Let $(e_n)_n$ be a k -spreading model of X which is equivalent to the usual basis of ℓ^1 . Also let $c = \inf \left\| \sum_{j=1}^n a_j e_j \right\|$, taken over all $n \in \mathbb{N}$ and $a_1, \dots, a_n \in \mathbb{R}$ with $\sum_{j=1}^n |a_j| = 1$. Let $\varepsilon > 0$ and choose $0 < \varepsilon' < c$, $p \in \mathbb{N}$ and b_1, \dots, b_p in $[-1, 1]$ with $\sum_{i=1}^p |b_i| = 1$ such that

$$(21) \quad \frac{c - \varepsilon'}{c + 2\varepsilon'} \geq 1 - \varepsilon \quad \text{and} \quad c \leq \left\| \sum_{i=1}^p b_i e_i \right\| \leq c + \varepsilon'$$

Let $(x_s)_{s \in [\mathbb{N}]^k}$ be a k -sequence in X generating $(e_n)_n$ as a k -spreading model. By passing to an infinite subset M of \mathbb{N} , we may assume that for every $n \in \mathbb{N}$, $a_1, \dots, a_n \in [-1, 1]$ and $(s_i)_{i=1}^n \in Plm_n([M]^k)$ with $s_1(1) \geq M(n)$, we have

$$(22) \quad \left| \left\| \sum_{i=1}^n a_i x_{s_i} \right\| - \left\| \sum_{i=1}^n a_i e_i \right\| \right| \leq \varepsilon' \sum_{i=1}^n |a_i|$$

Hence by (21), for every $(s_i)_{i=1}^p \in Plm_p([M]^k)$ with $s_1(1) \geq M(p)$ we have that

$$(23) \quad c - \varepsilon' \leq \left\| \sum_{i=1}^p b_i x_{s_i} \right\| \leq c + 2\varepsilon'$$

For every $s = (n_1, \dots, n_k) \in [\mathbb{N}]^k$, we set

$$(24) \quad y_s = \frac{\sum_{i=1}^p b_i x_{t_i^s}}{c + 2\varepsilon'} \text{ where } t_i^s = (M(p \cdot n_j + i - 1))_{j=1}^k, \text{ for all } 1 \leq i \leq p$$

Notice that $(t_i^s)_{i=1}^p \in Plm_p([\mathbb{N}]^k)$ and $t_1^s(1) = M(p \cdot s(1)) \geq M(p)$. Hence, by (21) and (23), it is clear that $1 - \varepsilon \leq \|y_s\| \leq 1$. Moreover, the k -subsequence $(y_s)_{s \in [\mathbb{N}]^k}$ generates ℓ^1 as a k -spreading model of constant $1 - \varepsilon$. Indeed, let $l \in \mathbb{N}$, $a_1, \dots, a_l \in [-1, 1]$ and $(s_j)_{j=1}^l \in Plm_l([\mathbb{N}]^k)$ with $s_1(1) \geq l$. Notice that $(t_i^{s_1})_{i=1}^p \cap \dots \cap (t_i^{s_l})_{i=1}^p \in Plm_{p \cdot l}([\mathbb{N}]^k)$ and $t_1^{s_1}(1) = M(p \cdot s_1(1)) \geq M(p \cdot l)$. Hence,

$$\left\| \sum_{j=1}^l a_j y_{s_j} \right\| = \left\| \sum_{j=1}^l a_j \cdot \sum_{i=1}^p \frac{b_i x_{t_i^{s_j}}}{c + 2\varepsilon'} \right\| \stackrel{(22)}{\geq} \frac{c - \varepsilon'}{c + 2\varepsilon'} \sum_{j=1}^l \sum_{i=1}^p |a_j| \cdot |b_i| \stackrel{(21)}{\geq} (1 - \varepsilon) \sum_{j=1}^l |a_j|$$

and the proof is complete. \square

Remark 11. If we additionally assume that X has a Schauder basis and $(x_s)_{s \in [M]^k}$ is plegma block (resp. plegma disjointly supported) then by (24) it is easy to see that $(y_s)_{s \in [L]^k}$ is also plegma block (resp. plegma disjointly supported).

8.3. Plegma block generated k -spreading models equivalent to the ℓ^1 basis. It well known that if a Banach space X with a Schauder basis admits an ℓ^1 spreading model, then X contains a block sequence which generates an ℓ^1 spreading model. In this subsection we extend this result. More precisely, we have the following.

Theorem 51. *Let X be a Banach space with a Schauder basis and $k \in \mathbb{N}$. Suppose that $\mathcal{SM}_k^{wrc}(X)$ contains up to equivalence the usual basis of ℓ^1 . Then there exists a plegma block generated k -spreading model of X equivalent to the usual basis of ℓ^1 .*

Proof. Let k_X be the minimum of all $k \in \mathbb{N}$ such that the set $\mathcal{SM}_k^{wrc}(X)$ contains a sequence equivalent to the usual basis of ℓ^1 . By Remark 8, it suffices to show that $\mathcal{SM}_{k_X}(X)$ contains a sequence equivalent to the usual basis of ℓ_1 which is plegma block generated. For $k_X = 1$ this is a well known standard fact. So suppose that $k_X = k \geq 2$ and let $(e_n)_n \in \mathcal{SM}_k^{wrc}(X)$ be equivalent to the usual basis of ℓ^1 . By Corollary 34, we may assume that $(e_n)_n$ is generated as a k -spreading model by a k -sequence $(x_s)_{s \in [\mathbb{N}]^k}$ which is subordinated with respect to the weak topology of X and admits a canonical tree decomposition $(y_t)_{t \in [\mathbb{N}]^{\leq k}}$.

Let $(w_v)_{v \in [\mathbb{N}]^{k-1}}$ be the $(k-1)$ -sequence in X defined by $w_v = \sum_{t \sqsubseteq v} y_t$, for every $v \in [\mathbb{N}]^{k-1}$. Also let $(x'_s)_{s \in [\mathbb{N}]^k}$, be the k -sequence defined by $x'_s = w_{s|k-1}$, for every $s \in [\mathbb{N}]^k$. Notice that $(w_v)_{v \in [\mathbb{N}]^{k-1}}$ is subordinated with respect to the weak topology. Hence $(w_v)_{v \in [\mathbb{N}]^{k-1}}$ is a weakly relatively compact $(k-1)$ -sequence. Also, by Lemma 22 we have that $(w_v)_{v \in [\mathbb{N}]^{k-1}}$ and $(x'_s)_{s \in [\mathbb{N}]^k}$ admit the same $(k-1)$ -spreading models. Therefore, since the usual basis of ℓ_1 is not contained up to equivalence in $\mathcal{SM}_{k-1}^{wrc}(X)$, we conclude that $(x'_s)_{s \in [\mathbb{N}]^k}$ does not admit a k -spreading model equivalent to the usual basis of ℓ^1 . Since $x_s = x'_s + y_s$, for all $s \in [\mathbb{N}]^k$, by Corollary 49, we get that the k -sequence $(y_s)_{s \in [\mathbb{N}]^k}$ admits a k -spreading model

equivalent to the usual basis of ℓ^1 . Since $(y_s)_{s \in [\mathbb{N}]^k}$ is a plegma block k -sequence in X (see Proposition 32 (iii)), the proof is complete. \square

8.4. Duality of c_0 and ℓ^1 k -spreading models. It is well known that if a Banach space X admits a c_0 spreading model, then X^* admits an ℓ^1 spreading model. In this subsection we extend this result.

Lemma 52. *Let X be a Banach space with a Schauder basis, $k \in \mathbb{N}$ and $(x_s)_{s \in [\mathbb{N}]^k}$ be a k -sequence in X which admits a canonical tree decomposition $(y_t)_{t \in [\mathbb{N}]^{\leq k}}$ and generates a k -spreading model equivalent to the usual basis of c_0 . Then $y_\emptyset = 0$ and there exist $1 \leq j_0 \leq k$ and $L \in [\mathbb{N}]^\infty$ such that the k -subsequence $(y_{s|j_0})_{s \in [L]^k}$ is plegma block and generates c_0 as a k -spreading model.*

Proof. Since $(x_s)_{s \in [\mathbb{N}]^k}$ generates a k -spreading model, we have that $(x_s)_{s \in [\mathbb{N}]^k}$ is seminormalized. Let (e_n) be the k -spreading model of $(x_s)_{s \in [\mathbb{N}]^k}$. Since $(e_n)_n$ is equivalent to the usual basis of c_0 , we have that $(e_n)_n$ is Cesaro summable to zero. Using these observations we may easily conclude that $y_\emptyset = 0$. We also observe that there exists $\delta > 0$ such that for every $s \in [\mathbb{N}]^k$ there exists $1 \leq j \leq k$ such that $\|y_{s|j}\| > \delta$. Hence by Ramsey's theorem there exists $1 \leq j_0 \leq k$ and $L \in [\mathbb{N}]^\infty$ such that for every $s \in [L]^k$, $\|y_{s|j_0}\| > \delta$.

Let $n \in \mathbb{N}$, $a_1, \dots, a_n \in \mathbb{R}$ and $(s_i)_{i=1}^n \in Plm_n([L]^k)$. If I is the interval of \mathbb{N} with $\min I = \min \text{supp}(y_{s_1|j_0})$ and $\max I = \max \text{supp}(x_{s_n|j_0})$, then Proposition 32 (v) and the fact that $y_\emptyset = 0$, yield that

$$I\left(\sum_{i=1}^n a_i x_{s_i}\right) = \sum_{i=1}^n a_i y_{s_i|j_0}$$

Hence if C is the basis constant of the Schauder basis of X , we get that

$$\frac{\delta}{2C} \max_{1 \leq i \leq n} |a_i| \leq \left\| \sum_{i=1}^n a_i y_{s_i|j_0} \right\| \leq 2C \left\| \sum_{i=1}^n a_i x_{s_i} \right\|$$

Therefore, since $(x_s)_{s \in [L]^k}$ generates c_0 as a k -spreading model, we conclude that every k -spreading model of $(y_{s|j_0})_{s \in [L]^k}$ is equivalent to the usual basis of c_0 . \square

The above lemma shows that the analogue of Theorem 51 for the c_0 basis also holds. Namely we have the following.

Corollary 53. *Let X be a Banach space with a Schauder basis and $k \in \mathbb{N}$. Suppose that $\mathcal{SM}_k^{wrc}(X)$ contains up to equivalence the usual basis of c_0 . Then there exists a plegma block generated k -spreading model of X equivalent to the usual basis of c_0 .*

Theorem 54. *Let X be a Banach space. Assume that for some $k \in \mathbb{N}$ the set $\mathcal{SM}_k^{wrc}(X)$ contains a sequence equivalent to the usual basis of c_0 . Then X^* admits ℓ^1 as a k -spreading model.*

Proof. Let $(x_s)_{s \in [\mathbb{N}]^k}$ be a subordinated k -sequence in X generating c_0 as a spreading model. Let Y separable subspace of X containing the k -sequence $(x_s)_{s \in [\mathbb{N}]^k}$ and $T : Y \rightarrow C[0, 1]$ an isometry. Notice that $C[0, 1]$ is a Banach space with a bimonotone Schauder basis and $(T(x_s))_{s \in [\mathbb{N}]^k}$ is subordinated. Let $(\varepsilon_n)_n$ a null sequence of positive reals. By Theorem 33 there exist $L \in [\mathbb{N}]^\infty$ and a k -subsequence $(\tilde{x}_s)_{s \in [L]^k}$ in $C[0, 1]$ satisfying the following.

(P1) $(\tilde{x}_s)_{s \in [L]^k}$ admits a canonical tree decomposition $(\tilde{y}_t)_{t \in [M]^{\leq k}}$.

(P2) For every $s \in [L]^k$, $\|T(x_s) - \tilde{x}_s\| < \varepsilon_n$, where $\min s = L(n)$.

Notice that property (P2) yields that $(\tilde{x}_s)_{s \in [L]^k}$ generates c_0 as a k -spreading model. By Lemma 52 there exist $M \in [L]^\infty$ and $1 \leq j_0 \leq k$ such that the plegma block k -subsequence $(\tilde{y}_{s|j_0})_{s \in [M]^k}$ generates c_0 as a k -spreading model. For every $s \in [M]^k$ we pick $\tilde{y}_s^* \in S_{C[0,1]^*}$ with $\tilde{y}_s^*(\tilde{y}_{s|j_0}) = \|\tilde{y}_{s|j_0}\|$ and $\text{supp } \tilde{y}_s^* \subseteq \text{range } \tilde{y}_{s|j_0}$. For every $s \in [M]^k$ we set $y_s^* = T^*(\tilde{y}_s^*)$ and we choose x_s^* in X^* an extension of y_s^* of the same norm. It is easy to check that $(x_s^*)_{s \in [M]^k}$ admits ℓ^1 as a spreading model. \square

9. k -CESÀRO SUMMABILITY VS ℓ^1 k -SPREADING MODELS

In this section we extend the well known dichotomy of H.P. Rosenthal concerning Cesàro summability and ℓ^1 spreading models (see also [4], [15]). We start by introducing the definition of the Cesàro summability for k -sequences in Banach spaces.

9.1. Definition of the k -Cesàro summability in Banach spaces.

Definition 55. Let X be a Banach space, $x_0 \in X$, $k \in \mathbb{N}$, $(x_s)_{s \in [\mathbb{N}]^k}$ be a k -sequence in X and $M \in [\mathbb{N}]^\infty$. We will say that the k -subsequence $(x_s)_{s \in [M]^k}$ is k -Cesàro summable to x_0 if

$$\binom{n}{k}^{-1} \sum_{s \in [M|n]^k} x_s \xrightarrow[n \rightarrow \infty]{\|\cdot\|} x_0$$

where $M|n = \{M(1), \dots, M(n)\}$.

Proposition 56. Let X be a Banach space, $x_0 \in X$, $k \in \mathbb{N}$, $(x_s)_{s \in [\mathbb{N}]^k}$ be a k -sequence in X and $M \in [\mathbb{N}]^\infty$.

- (i) If $(x_s)_{s \in [M]^k}$ norm converges to x_0 , then $(x_s)_{s \in [M]^k}$ is k -Cesàro summable to x_0 .
- (ii) If $(x_s)_{s \in [M]^k}$ is k -Cesàro summable to x_0 and in addition it is weakly convergent, then x_0 is the weak limit of $(x_s)_{s \in [M]^k}$.
- (iii) If X^* is separable and for every $N \in [M]^\infty$, $(x_s)_{s \in [N]^k}$ is k -Cesàro summable to x_0 then there exists $L \in [M]^\infty$ such that $(x_s)_{s \in [L]^k}$ weakly converges to x_0 .

Proof. Assertions (i) and (ii) are straightforward. For (iii), first observe that for every $x^* \in X^*$, $\varepsilon > 0$ and $N \in [M]^\infty$ there exists an $L \in [N]^\infty$ such that $|x^*(x_s) - x^*(x_0)| < \varepsilon$, for all $s \in [L]^k$. Next for a norm dense subset $\{x_n^* : n \in \mathbb{N}\}$ of X^* , we inductively choose an $L \in [M]^\infty$ such that for every $n \in \mathbb{N}$ and $s \in [L]^k$ with $\min s \geq L(n)$ we have that $|x_n^*(x_s) - x_n^*(x_0)| < \frac{1}{n}$ for all $1 \leq i = 1 \leq n$. This yields that $(x_s)_{s \in [L]^k}$ weakly converges to x_0 . \square

Remark 12. It is open if assertion (iii) of the above proposition remains valid without any restriction for X^* .

9.2. A density result for plegma families in $[\mathbb{N}]^k$. In this subsection we will present a density Ramsey result concerning plegma families. For its proof, we will need the deep theorem of H. Furstenberg and Y. Katznelson [8]. Actually, we shall use the following finite version of this theorem (see also [9]).

Theorem 57. Let $k \in \mathbb{N}$, F be a finite subset of \mathbb{Z}^k and $\delta > 0$. Then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, every subset \mathcal{A} of $\{1, \dots, n\}^k$ of size at least δn^k has a subset of the form $a + dF$ for some $a \in \mathbb{Z}^k$ and $d \in \mathbb{N}$.

Our density result for plegma families is the following.

Proposition 58. *Let $k, l \in \mathbb{N}$ and $\delta > 0$. Then there exists $N_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and every subset \mathcal{A} of $[\{1, \dots, n\}]^k$ of size at least $\delta \binom{n}{k}$, there exists a plegma family $(s_j)_{j=1}^l \in Plm_l([\mathbb{N}]^k)$ such that $s_j \in \mathcal{A}$, for every $1 \leq j \leq l$.*

Proof. For every $1 \leq j \leq l$, let $t_j = (j, l+j, 2l+j, \dots, (k-1)l+j)$. Clearly $(t_j)_{j=1}^l \in Plm_l([\mathbb{N}]^k)$. We set $F = \{\mathbf{0}\} \cup \{t_j : 1 \leq j \leq l\}$, where $\mathbf{0} = (0, \dots, 0)$ is the zero element of \mathbb{Z}^k . Fix $\delta > 0$. Since $\lim_n \binom{n}{k}/n^k = 1/k!$, there exists $m_0 \in \mathbb{N}$ such that for every $n \geq m_0$ and every subset \mathcal{A} of $[\{1, \dots, n\}]^k$ of size at least $\delta \binom{n}{k}$ has density at least $\frac{\delta}{2k!}$ in $\{1, \dots, n\}^k$. Hence, by Theorem 57 (applied for $\frac{\delta}{2k!}$ in place of δ) we have that there exists $n_0 \geq m_0$ such that for every $n \geq n_0$, every subset \mathcal{A} of $[\{1, \dots, n\}]^k$ of size at least $\delta \binom{n}{k}$ has a subset of the form $a + dF$ for some $a \in \mathbb{Z}^k$ and $d \in \mathbb{N}$. Notice that $a = a + d\mathbf{0} \in \mathcal{A}$ and therefore $a \in [\{1, \dots, n\}]^k$. For every $j \in \{1, \dots, l\}$, we set $s_j = a + dt_j$. Then $\{s_j : 1 \leq j \leq l\} \subseteq \mathcal{A}$. Moreover, since $a \in [\mathbb{N}]^k$ and $d \in \mathbb{N}$, we easily conclude that $(s_j)_{j=1}^l \in Plm_l([\mathbb{N}]^k)$ and the proof is complete. \square

Remark 13. It is easy to see that for $k = 1$ the preceding lemma trivially holds (it suffices to set $N_0 = \lceil \frac{l}{\delta} \rceil$) and therefore Theorem 57 is actually used for $k \geq 2$. However, it is not completely clear to us if the full strength of such a deep theorem like Furstenberg-Katznelson's is actually necessary for the proof of Proposition 58.

9.3. The main results.

Proposition 59. *Let X be a Banach space, $k \in \mathbb{N}$ and $(x_s)_{s \in [\mathbb{N}]^k}$ be a bounded k -sequence in X . Let $M \in [\mathbb{N}]^\infty$ such that the subsequence $(x_s)_{s \in [M]^k}$ generates a Cesàro summable to zero k -spreading model $(e_n)_n$. Then for every $L \in [M]^\infty$ the k -subsequence $(x_s)_{s \in [L]^k}$ is k -Cesàro summable to zero.*

Proof. Assume on the contrary that there exists $L \in [M]^\infty$ such that $(x_s)_{s \in [L]^k}$ is not k -Cesàro summable to zero. Then there exists a $\theta > 0$ and a strictly increasing sequence $(p_n)_n$ of natural numbers such that for every $n \in \mathbb{N}$,

$$(25) \quad \binom{p_n}{k}^{-1} \left\| \sum_{s \in [L|p_n]^k} x_s \right\| > \theta$$

For each $n \in \mathbb{N}$, we pick $\bar{x}_n^* \in S_{X^*}$ such that $\bar{x}_n^* \left(\binom{p_n}{k}^{-1} \sum_{s \in [L|p_n]^k} x_s \right) > \theta$ and we set

$$(26) \quad \mathcal{A}_n = \left\{ s \in [\{1, \dots, p_n\}]^k : \bar{x}_n^*(x_{L(s)}) > \frac{\theta}{2} \right\}$$

where S_{X^*} is the unit sphere of X^* . By (25) and a simple averaging argument we easily derive that $|\mathcal{A}_n| \geq \frac{\theta}{2K} \binom{p_n}{k}$, where $K = \sup\{\|x_s\| : s \in [\mathbb{N}]^k\}$.

We fix $m \in \mathbb{N}$. By Proposition 58, with $\delta = \frac{\theta}{2K}$ and $l = 2m - 1$, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ there exists a plegma family $(s_j)_{j=1}^l \in Plm_l([\mathbb{N}]^k)$ such that $\{s_j : 1 \leq j \leq l\} \subseteq \mathcal{A}_n$. Therefore setting $t_i = L(s_{m+i-1})$ for all $1 \leq i \leq m$, we conclude that for every $m \in \mathbb{N}$ there exists $(t_i)_{i=1}^m \in Plm_m([L]^k)$ such that $t_1(1) \geq L(m)$ and $\left\| \frac{1}{m} \sum_{j=1}^m x_{t_j} \right\| > \frac{\theta}{2}$. This easily yields that $(e_n)_n$ is not Cesàro summable to zero, which is a contradiction. \square

Corollary 60. *Let X be a Banach space, $k \in \mathbb{N}$ and $(x_s)_{s \in [\mathbb{N}]^k}$ be a bounded k -sequence in X . Let $M \in [\mathbb{N}]^\infty$ such that the subsequence $(x_s)_{s \in [M]^k}$ generates an unconditional k -spreading model $(e_n)_n$. Then at least one of the following holds:*

- (1) *The sequence $(e_n)_n$ is equivalent to the usual basis of ℓ^1 .*
- (2) *For every $L \in [M]^\infty$ $(x_s)_{s \in [L]^k}$ is k -Cesàro summable to zero.*

Proof. Assume that $(e_n)_n$ is not equivalent to the usual basis of ℓ^1 . Since $(e_n)_n$ is an unconditional spreading sequence, by Proposition 14 we have that $(e_n)_n$ is Cesàro summable to zero. Hence, by Proposition 59 we have that $(x_s)_{s \in [L]^k}$ is k -Cesàro summable to zero, for every $L \in [M]^\infty$. \square

Remark 14. Notice that in the case $k = 1$ the two alternatives of Corollary 60 are mutually exclusive. This does not remain valid for $k \geq 2$. For instance, assume that in Example 1, $(e_n)_n$ is the usual basis of ℓ^1 . Then the basis $(x_s)_{s \in [\mathbb{N}]^{k+1}}$ of X generates a $(k+1)$ -spreading model equivalent to the usual basis of ℓ^1 and simultaneously for every $L \in [\mathbb{N}]^\infty$, $(x_s)_{s \in [L]^{k+1}}$ is $(k+1)$ -Cesàro summable to zero. Indeed, let $L \in [\mathbb{N}]^\infty$ and $n \in \mathbb{N}$. Then since every plegma tuple in $[L|n]^{k+1}$ is of size less than n , we have

$$\left\| \binom{n}{k+1}^{-1} \sum_{s \in [L|n]^{k+1}} x_s \right\|_{k+1} \leq n \binom{n}{k+1}^{-1}$$

Since $k+1 \geq 2$, $\lim_n n \binom{n}{k+1}^{-1} = 0$. Thus for every $L \in [\mathbb{N}]^\infty$, $(x_s)_{s \in [L]^{k+1}}$ is Cesàro summable to zero.

Theorem 61. *Let X be a Banach space, $k \in \mathbb{N}$ and $(x_s)_{s \in [\mathbb{N}]^k}$ be a weakly relatively compact k -sequence in X . Then there exists $M \in [\mathbb{N}]^\infty$ such that at least one of the following holds:*

- (1) *The subsequence $(x_s)_{s \in [M]^k}$ generates a k -spreading model equivalent to the usual basis of ℓ^1 .*
- (2) *There exists $x_0 \in X$ such that for every $L \in [M]^\infty$, $(x_s)_{s \in [L]^k}$ is k -Cesàro summable to x_0 .*

Proof. First we notice that if there exists $M \in [\mathbb{N}]^\infty$ such that $(x_s)_{s \in [M]^k}$ norm converges to some $x_0 \in X$, then by Proposition 56 (i), we immediately get that (2) holds. So we may suppose for the sequel that the k -sequence $(x_s)_{s \in [\mathbb{N}]^k}$ does not contain any norm convergent k -subsequence.

Let $M_1 \in [\mathbb{N}]^\infty$ such that $(x_s)_{s \in [M_1]^k}$ generates a k -spreading model $(e_n)_n$. By Proposition 27 there exists $M_2 \in [M_1]^\infty$ such that $(x_s)_{s \in [M_2]^k}$ is subordinated (with respect to the weak topology). Let $\widehat{\varphi} : [M_2]^k \rightarrow (X, w)$ be the continuous map witnessing this and $x_0 = \widehat{\varphi}(\emptyset)$.

For every $s \in [M_2]^k$ we set $x'_s = x_s - x_0$. Notice that the map $\widehat{\psi} : [M_2]^k \rightarrow (X, w)$ defined by $\widehat{\psi}(t) = \widehat{\varphi}(t) - x_0$ is continuous. Hence $(x'_s)_{s \in [M_2]^k}$ is subordinated. Since $\widehat{\psi}(\emptyset) = 0$, by Proposition 26, we have that $(x'_s)_{s \in [M_2]^k}$ is weakly null. Moreover, since $(x_s)_{s \in [\mathbb{N}]^k}$ does not contain any norm convergent k -subsequence, it is easy to see that $(x'_s)_{s \in [M_2]^k}$ is seminormalized.

Let $(e'_n)_n$ be a k -spreading model of $(x'_s)_{s \in [M_2]^k}$ and let $M \in [M_2]^\infty$ such that $(x'_s)_{s \in [M]^k}$ generates $(e'_n)_n$. By Theorem 36, $(e'_n)_n$ is unconditional and therefore, by Corollary 60, we have that either $(e'_n)_n$ is equivalent to the usual basis of ℓ^1 or for every $L \in [M]^\infty$, $(x'_s)_{s \in [L]^k}$ is k -Cesàro summable to zero. Since $x_s = x'_s + x_0$,

for every $s \in [M]^k$, by Lemma 37 we have that the first alternative yields that $(e_n)_n$ is equivalent to the usual basis of ℓ^1 while the second one, easily gives that for every $L \in [M]^\infty$, $(x_s)_{s \in [L]^k}$ is k -Cesàro summable to x_0 . \square

10. THE k -SPREADING MODELS OF c_0 AND ℓ^p , $1 \leq p < \infty$

In this section we deal with a natural problem, posed to us by Th. Schlumprecht, of determining the spreading models of the classical sequence spaces. As we will see, while the spreading models of ℓ^p , $1 \leq p < \infty$, are as expected, the class of the 2-spreading models of c_0 is surprising large.

10.1. The k -spreading models of c_0 . It is well known that every non trivial spreading model of c_0 generates a space isomorphic to c_0 . On the other hand the class of the 2-spreading models of c_0 is quite large. As we will see $\mathcal{SM}_2(c_0)$ contains all bimonotone Schauder basic spreading sequences. Notice that this property of c_0 is similar to the one of $C(\omega^\omega)$ admitting every 1-unconditional spreading sequence as a spreading model (see [16]).

We start with the following lemma.

Lemma 62. *Let $(e_n)_n$ be a spreading sequence in ℓ^∞ and let $(x_s)_{s \in [\mathbb{N}]^2}$ be the 2-sequence in c_0 defined by $x_s = (e_{s(1)}(1), e_{s(1)}(2), \dots, e_{s(1)}(s(2)), 0, 0, \dots)$, for every $s \in [\mathbb{N}]^2$. Then for every non trivial 2-spreading model $(\tilde{e}_n)_n$ of $(x_s)_{s \in [\mathbb{N}]^2}$, $l \in \mathbb{N}$ and $a_1, \dots, a_l \in \mathbb{R}$, we have*

$$(27) \quad \left\| \sum_{i=1}^l a_i e_i \right\|_\infty \leq \left\| \sum_{i=1}^l a_i \tilde{e}_i \right\| \leq \max_{1 \leq j \leq l} \left\| \sum_{i=j}^l a_i e_i \right\|_\infty$$

Proof. We fix $l \in \mathbb{N}$ and $a_1, \dots, a_l \in \mathbb{R}$. It is easy to check that for every $(s_i)_{i=1}^l \in Plm_l([\mathbb{N}]^k)$, we have that

$$(28) \quad \left\| \sum_{i=1}^l a_i x_{s_i} \right\|_\infty \leq \max_{1 \leq j \leq l} \left\| \sum_{i=j}^l a_i e_{s_i(1)} \right\|_\infty$$

Let $M \in [\mathbb{N}]^\infty$ such that $(x_s)_{s \in [M]^2}$ generates a non trivial 2-spreading model $(\tilde{e}_n)_n$. Then by (28), we easily obtain the righthand inequality of (27). To complete the proof, we fix $\varepsilon > 0$ and $m_\varepsilon \in \mathbb{N}$ such that

$$(29) \quad \left\| \sum_{i=1}^l a_i e_i \right\|_\infty - \varepsilon \leq \left| \sum_{i=1}^l a_i e_i(m_\varepsilon) \right|$$

Notice that for every $(s_i)_{i=1}^l \in Plm_l([\mathbb{N}]^2)$ in $[\mathbb{N}]^2$ with $s_1(1) \geq m$, we have that

$$(30) \quad \left| \sum_{i=1}^l a_i e_i(m) \right| \leq \left\| \sum_{i=1}^l a_i x_{s_i} \right\|_\infty$$

Therefore, since $(x_s)_{s \in [M]^2}$ generates $(\tilde{e}_n)_n$ as a 2-spreading model, by (29) and (30), we get that

$$\left\| \sum_{i=1}^l a_i e_i \right\|_\infty - \varepsilon \leq \left| \sum_{i=1}^l a_i e_i(m_\varepsilon) \right| \leq \left\| \sum_{i=1}^l a_i \tilde{e}_i \right\|_\infty$$

Since this holds for every $\varepsilon > 0$, we obtain the lefthand inequality of (27) and the proof is complete. \square

Proposition 63. *For every Schauder basic spreading sequence $(e_n)_n$ there exists $(\tilde{e}_n)_n \in \mathcal{SM}_2(c_0)$ equivalent to $(e_n)_n$. In particular, if $(e_n)_n$ is bimonotone then $(e_n)_n$ is contained in $\mathcal{SM}_2(c_0)$.*

Proof. We may assume that $(e_n)_n$ is a sequence in ℓ^∞ . Let $C > 0$ be the basis constant of $(e_n)_n$. By Lemma 62 there exists $(\tilde{e}_n)_n \in \mathcal{SM}_2(c_0)$ such that for all $l \in \mathbb{N}$ and $a_1, \dots, a_l \in \mathbb{R}$, we have

$$(31) \quad \left\| \sum_{i=1}^l a_i e_i \right\|_\infty \leq \left\| \sum_{i=1}^l a_i \tilde{e}_i \right\| \leq \max_{1 \leq j \leq l} \left\| \sum_{i=j}^l a_i e_i \right\|_\infty \leq (1+C) \left\| \sum_{i=1}^l a_i e_i \right\|_\infty$$

Hence, $(e_n)_n$ and $(\tilde{e}_n)_n$ are equivalent. Moreover, if in addition $(e_n)_n$ is bimonotone then $\max_{1 \leq j \leq l} \left\| \sum_{i=j}^l a_i e_i \right\|_\infty \leq \left\| \sum_{i=1}^l a_i e_i \right\|_\infty$ and therefore $(\tilde{e}_n)_n$ is isometric to $(e_n)_n$. \square

Corollary 64. *For every singular spreading sequence $(e_n)_n$, there exists $(\tilde{e}_n)_n \in \mathcal{SM}_2(c_0)$ equivalent to $(e_n)_n$.*

Proof. Let $e_n = e'_n + e$ be the natural decomposition of $(e_n)_n$. By Remark 5, $(e'_n)_n$ is spreading and 1-unconditional. Hence, by Proposition 63, there exists a 2-sequence $(x_s)_{s \in [\mathbb{N}]^2}$ in c_0 generating $(e'_n)_n$ as a 2-spreading model. For every $s \in [\mathbb{N}]^2$, let \tilde{x}_s be the sequence in c_0 defined by $\tilde{x}_s(1) = \|e\|$ and $\tilde{x}_s(n+1) = x_s(n)$ for all $n \in \mathbb{N}$. It is easy to see that $(\tilde{x}_s)_{s \in [\mathbb{N}]^2}$ generates a 2-spreading model $(\tilde{e}_n)_n$, satisfying

$$\left\| \sum_{j=1}^n a_j \tilde{e}_j \right\| = \max \left\{ \left\| \sum_{j=1}^n a_j \right\| \cdot \|e\|, \left\| \sum_{j=1}^n a_j e'_j \right\| \right\}$$

for all $n \in \mathbb{N}$ and $a_1, \dots, a_n \in \mathbb{R}$. Therefore, by Remark 5, we conclude that $(e_n)_n$ and $(\tilde{e}_n)_n$ are equivalent. \square

By Proposition 63 and Corollary 64 we have the following.

Corollary 65. *The set $\mathcal{SM}_2(c_0)$ is isomorphically universal for all spreading sequences.*

10.2. The k -spreading models of ℓ^p , for $1 \leq p < \infty$. The k -spreading models of the spaces ℓ^p , for $1 \leq p < \infty$, can be treated as the classical spreading models. This is based on the observation that the usual basis of these spaces is symmetric. Therefore, the norm-behavior of the k -sequences admitting a canonical tree decomposition is identical with the one of sequences being of the form $(x_n + x)_n$, where $(x_n)_n$ is block.

Especially, for the case of ℓ^1 , one has to make use of the w^* -relative compactness of the bounded k -sequences in order to pass to a subordinated k -subsequence with respect to the w^* -topology and in turn to a further one which is approximated by a k -subsequence admitting a canonical tree decomposition. This procedure yields the following.

Theorem 66. *Let $1 \leq p < \infty$ and $(\tilde{e}_n)_n$ be a k -spreading model of ℓ^p , for some $k \in \mathbb{N}$. Then there exist $a_1, a_2 \geq 0$ such that $(\tilde{e}_n)_n$ is isometric to the sequence $(a_1 e_1 + a_2 e_{n+1})_n$, where $(e_n)_n$ denotes the usual basis of ℓ^p . More precisely we have the following.*

- (i) *The sequence $(\tilde{e}_n)_n$ is trivial if and only if $a_2 = 0$.*
- (ii) *The sequence $(\tilde{e}_n)_n$ is singular if and only if $a_1 \neq 0$ and $a_2 \neq 0$.*

- (iii) The sequence $(\tilde{e}_n)_n$ is Schauder basic if and only if $a_1 = 0$ and $a_2 \neq 0$. In this case $(\tilde{e}_n)_n$ is equivalent to the usual basis of ℓ^p .

Remark 15. It can also be shown that every $(\tilde{e}_n)_n \in \mathcal{SM}_k^{wrc}(c_0)$, satisfies the analogue of Theorem 66 with c_0 in place of ℓ^p .

Corollary 67. Every non trivial k -spreading model of ℓ^p , $1 < p < \infty$, generates a space isometric to ℓ^p . In particular, every non trivial k -spreading model of ℓ^1 is Schauder basic and equivalent to the usual basis of ℓ^1 .

11. A REFLEXIVE SPACE NOT ADMITTING ℓ^p OR c_0 AS A SPREADING MODEL

A space not admitting any ℓ^p , for $1 \leq p < \infty$, or c_0 spreading model was constructed in [17]. In the same paper it is asked if there exists a space which does not contain any ℓ^p , for $1 \leq p < \infty$, or c_0 k -iterated spreading model of any $k \in \mathbb{N}$. In this section we give an example of a reflexive space X answering affirmatively this problem.

11.1. The definition of the space X . The construction of X is closely related to the corresponding one in [17]. Let $(n_j)_j$ and $(m_j)_j$ be two strictly increasing sequences of natural numbers satisfying the following:

- (i) $\sum_{j=1}^{\infty} \frac{1}{m_j} \leq 0, 1$.
- (ii) For every $a > 0$, we have that $\frac{n_j^a}{m_j} \xrightarrow{j \rightarrow \infty} \infty$.
- (iii) For every $j \in \mathbb{N}$, we have that $\frac{n_j}{n_{j+1}} < \frac{1}{m_j}$.

Let $\|\cdot\|$ be the norm on $c_{00}(\mathbb{N})$, implicitly defined as follows. For every $x \in c_{00}(\mathbb{N})$ we set

$$(32) \quad \|x\| = \max \left\{ \|x\|_{\infty}, \left(\sum_{j=1}^{\infty} \|x\|_j^2 \right)^{\frac{1}{2}} \right\}$$

where $\|x\|_j = \sup \left\{ \frac{1}{m_j} \sum_{q=1}^{n_j} \|E_q(x)\| : E_1 < \dots < E_{n_j} \right\}$.

Let X be the completion of $c_{00}(\mathbb{N})$ under the above norm. It is easy to see that the Hamel basis of $c_{00}(\mathbb{N})$ is an unconditional basis of the space X . Also notice that for every $x \in X$ the sequence $w = (\|x\|_j)_j$ belongs to ℓ^2 and $\left(\sum_{j=1}^{\infty} \|x\|_j^2 \right)^{\frac{1}{2}} = \|w\|_{\ell^2} \leq \|x\|$.

11.2. The main results. The following is the main result of this section.

Theorem 68. For every $k \in \mathbb{N}$ and $(e_n)_n \in \mathcal{SM}_k(X)$, the space E generated by $(e_n)_n$ does not contain any isomorphic copy of ℓ^p , $1 \leq p < \infty$, or c_0 .

Given the above theorem we get the following consequence which the aforementioned problem stated in [17].

Corollary 69. For every $k \in \mathbb{N}$, the spaces generated by the k -iterated spreading models of X do not contain any isomorphic copy of ℓ^p , $1 \leq p < \infty$, or c_0 .

Proof. By Theorem 68 and James' Theorem we have that for every $k \in \mathbb{N}$, the spaces generated by the unconditional k -spreading models of X are reflexive. By Corollary 47 we have that for every $k \in \mathbb{N}$, every space generated by a k -iterated spreading model of X is isomorphic to the space generated by an unconditional k -spreading model of X . By Theorem 68, the proof is complete. \square

Also notice this example shows that Krivine's theorem [13] concerning ℓ^p or c_0 block finite representability cannot be captured by the notion of k -spreading models.

11.3. Proof of Theorem 68. We will need the next well known lemma (see [4]).

Lemma 70. *Let $j < j_0$ in \mathbb{N} and $(x_q)_{q=1}^{n_{j_0}}$ be a block sequence in the unit ball B_X of X . Then*

$$\left\| \frac{x_1 + \dots + x_{n_{j_0}}}{n_{j_0}} \right\|_j < \frac{2}{m_j}$$

Lemma 71. *Let $d_0 < j_0$ in \mathbb{N} , and $(x_q)_{q=1}^{n_{j_0}}$ be a block sequence in B_X . We set $E = \{n \in \mathbb{N} : n > d_0\}$ and $w_q = (\|x_q\|_j)_{j \in E}$, for all $1 \leq q \leq n_{j_0}$. Assume that for some $0 < \varepsilon < 1$ there exists a disjointly supported finite sequence $(w'_q)_{q=1}^{n_{j_0}}$ in ℓ^2 such that $\|E(w_q - w'_q)\|_{\ell^2} < \varepsilon$, for all $1 \leq q \leq n_{j_0}$. Then*

$$\left\| \frac{x_1 + \dots + x_{n_{j_0}}}{n_{j_0}} \right\| < 0.2 + \varepsilon + 2n_{j_0}^{-\frac{1}{2}}$$

Proof. By Lemma 70, we have that

$$\left\| \left(\left\| \frac{\sum_{q=1}^{n_{j_0}} x_q}{n_{j_0}} \right\|_j \right)_{j=1}^{d_0} \right\|_{\ell^2} \leq \sum_{j=1}^{d_0} \left\| \frac{\sum_{q=1}^{n_{j_0}} x_q}{n_{j_0}} \right\|_j \leq \sum_{j=1}^{d_0} \frac{2}{m_j} < 0.2$$

Using the above and the observation that $\|E(w'_q)\|_{\ell^2} \leq 2$, for all $1 \leq q \leq n_{j_0}$, we get the following.

$$\begin{aligned} \left\| \left(\left\| \frac{1}{n_{j_0}} \sum_{q=1}^{n_{j_0}} x_q \right\|_j \right)_j \right\|_{\ell^2} &\leq 0.2 + \left\| \left(\left\| \frac{1}{n_{j_0}} \sum_{q=1}^{n_{j_0}} x_q \right\|_j \right)_{j>d_0} \right\|_{\ell^2} \\ &\leq 0.2 + \left\| \frac{1}{n_{j_0}} \sum_{q=1}^{n_{j_0}} (w_q(j))_{j>d_0} \right\|_{\ell^2} \leq 0.2 + \left\| \sum_{q=1}^{n_{j_0}} \frac{E(w'_q)}{n_{j_0}} \right\|_{\ell^2} + \varepsilon \\ &\leq 0.2 + \left(\sum_{q=1}^{n_{j_0}} \left(\frac{2}{n_{j_0}} \right)^2 \right)^{\frac{1}{2}} + \varepsilon = 0.2 + \varepsilon + 2n_{j_0}^{-\frac{1}{2}} \end{aligned}$$

Moreover $\left\| \frac{1}{n_{j_0}} \sum_{q=1}^{n_{j_0}} x_q \right\|_{\infty} \leq \frac{1}{n_{j_0}} < \frac{1}{m_1} < 0.1$. Hence by (32) the proof is completed. \square

Lemma 72. *For all $k \in \mathbb{N}$, every plegma block generated k -spreading model of X is not equivalent to the usual basis of ℓ^1 .*

Proof. Assume on the contrary that there exist $k \in \mathbb{N}$ and a plegma block k -sequence $(x_s)_{s \in [\mathbb{N}]^k}$ in X which generates ℓ^1 as a k -spreading model. By Proposition 50, we may also assume that $x_s \in B_X$, for all $s \in [\mathbb{N}]^k$ and $(x_s)_{s \in [\mathbb{N}]^k}$ generates ℓ^1 as a k -spreading model of constant $1 - \varepsilon$, where $\varepsilon = 0, 1$.

For every $s \in [\mathbb{N}]^k$, let $w_s = (\|x_s\|_j)_{j \in E}$. Since $(w_s)_{s \in [\mathbb{N}]^k}$ is a k -sequence in B_{ℓ^2} , it is weakly relatively compact. Hence, by Proposition 29, there exists $M \in [\mathbb{N}]^{\infty}$ such that the k -subsequence $(w_s)_{s \in [M]^k}$ is subordinated with respect the weak topology on ℓ^2 . Let $\hat{\varphi} : [M]^{\leq k} \rightarrow (\ell^2, w)$ be the continuous map witnessing this. By Theorem 33, there exist $L \in [M]^{\infty}$ and a k -subsequence $(\tilde{w}_s)_{s \in [L]^k}$ in X satisfying the following.

- (i) $(\tilde{w}_s)_{s \in [L]^k}$ admits a canonical tree decomposition $(\tilde{z}_t)_{t \in [L]^{\leq k}}$ with $\tilde{z}_{\emptyset} = \hat{\varphi}(\emptyset)$.

- (ii) For every $s \in [L]^k$, $\|w_s - \tilde{w}_s\|_{\ell^2} < \varepsilon/2$, where $\min s = L(n)$.
- (iii) The k -subsequence $(\tilde{w}_s)_{s \in [L]^k}$ is subordinated with respect to the weak topology of ℓ^2 .

Let $d_0 \in \mathbb{N}$ such that $\|E(\widehat{\varphi}(\emptyset))\|_{\ell^2} < \frac{\varepsilon}{2}$, where $E = \{d_0 + 1, \dots\}$. For every $s \in [L]^k$ we set $w'_s = \tilde{w}_s - \widehat{\varphi}(\emptyset)$. By Proposition 32 (iv), we have that $(w'_s)_{s \in [L]^k}$ is plegma disjointly supported. Moreover, notice that $\|E(w_s - w'_s)\|_{\ell^2} < \varepsilon$, for all $s \in [L]^k$. We pick $j_0 > d_0$ such that $2n_{j_0}^{-\frac{1}{2}} < \varepsilon$. Since $(x_s)_{s \in [\mathbb{N}]^k}$ generates ℓ^1 as a k -spreading model of constant 0, 9, we may choose $(s_q)_{q=1}^{n_{j_0}} \in \text{Plm}_{n_{j_0}}([L]^k)$ such that

$$(33) \quad \left\| \frac{1}{n_{j_0}} \sum_{q=1}^{n_{j_0}} x_{s_q} \right\| \geq 0,8$$

Observe that $d_0, j_0, \varepsilon, (x_{s_q})_{q=1}^{n_{j_0}}$ and $(w'_{s_q})_{q=1}^{n_{j_0}}$ satisfy the assumptions of Lemma 71. Hence

$$\left\| \frac{1}{n_{j_0}} \sum_{q=1}^{n_{j_0}} x_{s_q} \right\| < 0,2 + \varepsilon + 2n_{j_0}^{-\frac{1}{2}} < 0,4$$

which contradicts (33) and the proof is complete. \square

Corollary 73. *The space X is reflexive.*

Proof. Lemma 72 implies that the space X does not contain any isomorphic copy of ℓ^1 . Moreover, using that $\frac{n_j}{m_j} \xrightarrow{j \rightarrow \infty} \infty$, it is easy to see that the space X does not contain any isomorphic copy of c_0 . Since the basis of X is unconditional, the result follows by James' theorem. \square

Corollary 74. *For all $k \in \mathbb{N}$, every k -spreading model of X is not equivalent to the usual basis of ℓ^1 .*

Proof. Suppose on the contrary that there exist $k \in \mathbb{N}$ and a bounded k -sequence $(x_s)_{s \in [\mathbb{N}]^k}$ which generates a k -spreading model equivalent to the ℓ^1 basis. By the reflexivity of X , we have that $(x_s)_{s \in [\mathbb{N}]^k}$ is weakly relatively compact. Therefore, by Theorem 51, there exists a plegma block generated k -spreading model of X equivalent to the usual basis of ℓ^1 , which contradicts to Lemma 72. \square

Lemma 75. *Let $1 < p \leq \infty$. Then for every $\delta, C > 0$ there exists $l_0 \in \mathbb{N}$ such that for every $l \geq l_0$ and every block sequence $(x_q)_{q=1}^{n_l}$ in X with $\|x_q\| > \delta$, for all $1 \leq q \leq n_l$, we have that*

$$\left\| \sum_{q=1}^{n_l} x_q \right\| > C n_l^{\frac{1}{p}}$$

where by convention $\frac{1}{\infty} = 0$

Proof. Since $\frac{n_l}{m_l} \xrightarrow{l \rightarrow \infty} \infty$, there exists $l_0 \in \mathbb{N}$ such that $\frac{n_l}{m_l} > \frac{C}{\delta}$, for every $l \geq l_0$. Let $(x_q)_{q=1}^{n_l}$ be a block sequence in X with $\|x_q\| > \delta$, for all $1 \leq q \leq n_l$. Then

$$\left\| \sum_{q=1}^{n_l} x_q \right\| \geq \left\| \sum_{q=1}^{n_l} x_q \right\|_l \geq \frac{1}{m_l} \sum_{q=1}^{n_l} \|x_q\| > \frac{n_l}{m_l} \delta > C n_l^{\frac{1}{p}}$$

\square

Corollary 76. *For all $k \in \mathbb{N}$, every k -spreading model of X is not equivalent to the usual basis of ℓ^p , $1 < p < \infty$, or c_0 .*

Proof. Suppose on the contrary that for some $k \in \mathbb{N}$, X admits a k -spreading model $(e_n)_n$, which is equivalent to the usual basis of either ℓ^p , for some $1 < p < \infty$, or c_0 . First we shall treat the case of ℓ^p . Since X is reflexive, we have that $(e_n)_n \in \mathcal{SM}_k^{wrc}(X)$. By Corollary 34, there exists a subordinated k -sequence $(x_s)_{s \in [\mathbb{N}]^k}$ admitting a canonical tree decomposition $(y_t)_{t \in [\mathbb{N}]^{\leq k}}$, which generates $(e_n)_n$ as a k -spreading model. Since the basis of X is unconditional and $(e_n)_n$ is Cesàro summable to zero, it is easy to see that $y_\emptyset = 0$. Notice that $(x_s)_{s \in [\mathbb{N}]^k}$ is seminormalized and let $\delta > 0$ such that $\|x_s\| > \delta$, for all $s \in [\mathbb{N}]^k$. Hence, for every $s \in [\mathbb{N}]^k$ there exists $1 \leq d \leq k$ such that $\|y_{s|d}\| > \frac{\delta}{k}$. By Ramsey's theorem there exists $1 \leq d \leq k$ and $L \in [\mathbb{N}]^\infty$ such that for every $s \in [L]^k$, $\|y_{s|d}\| > \frac{\delta}{k}$. By Proposition 32 (iii), we have that $(y_{s|d})_{s \in [L]^k}$ is plegma block. Fix $C > 0$. By Lemma 75 we have that there exists l_0 such that for every $l > l_0$ and $(s_q)_{q=1}^{n_l} \in Plm_{n_l}[L]^k$ we have that $\left\| \sum_{q=1}^{n_l} y_{s_q|d} \right\| > Cn_l^{\frac{1}{p}}$. Hence, by the 1-unconditionality of the basis of X , we conclude that

$$\left\| \sum_{q=1}^{n_l} x_{s_q} \right\| > Cn_l^{\frac{1}{p}}$$

Since the above holds for every $C > 0$ we have that $(e_n)_n$ is not equivalent to the usual basis of ℓ^p , which is a contradiction.

Finally, if $(e_n)_n$ is equivalent to the usual basis of c_0 , then the proof is carried out using identical arguments as above and applying Lemma 75 for $p = \infty$. \square

Proof of Theorem 68. Suppose that for some $k \in \mathbb{N}$ there exists $(e_n)_n \in \mathcal{SM}_k(X)$ such that the space E generated by $(e_n)_n$ contains an isomorphic copy of Y , where Y is either ℓ^p , for some $1 \leq p < \infty$, or c_0 . Obviously $(e_n)_n$ is non trivial. Since X is reflexive, $(e_n)_n \in \mathcal{SM}_k^{wrc}(X)$. By Corollary 43, we have that $\mathcal{SM}_{k+1}(X)$ contains a sequence equivalent to the usual basis of Y . By Corollaries 74 and 76, we get the contradiction. \square

12. A SPACE X SUCH THAT $\mathcal{SM}_k(X)$ IS A PROPER SUBSET OF $\mathcal{SM}_{k+1}(X)$

In this section we shall present a Banach space \mathfrak{X}_{k+1} , having an unconditional basis $(e_s)_{s \in [\mathbb{N}]^{k+1}}$ which generates a $(k+1)$ -spreading model equivalent to the usual basis of ℓ^1 , while the space \mathfrak{X}_{k+1} does not admit ℓ^1 as a k -spreading model. Moreover, $(e_s)_{s \in [\mathbb{N}]^{k+1}}$ is not $(k+1)$ -Cesàro summable to any x_0 in \mathfrak{X}_{k+1} .

12.1. The definition of the space \mathfrak{X}_{k+1} . We fix for the following a positive integer k . We will need the next definition.

Definition 77. *A family $\mathcal{P} \subseteq [\mathbb{N}]^{k+1}$ will be called plegmatic in $[\mathbb{N}]^{k+1}$, if there exist a finite block sequence $F_1 < \dots < F_{k+1}$ of subsets of \mathbb{N} with $|F_1| = \dots = |F_{k+1}|$ such that $\mathcal{P} \subseteq F_1 \times \dots \times F_{k+1}$. A plegmatic family $\mathcal{P} \subseteq [\mathbb{N}]^{k+1}$ will be called Schreier if in addition $|F_1| \leq \min F_1$.*

For instance, for every $(s_j)_{j=1}^l \in Plm_l(\mathbb{N})^{k+1}$, the family $\mathcal{P} = \{s_1, \dots, s_l\}$ is plegmatic but notice that not all plegmatic families in $[\mathbb{N}]^{k+1}$ are plegma.

Let $(e_s)_{s \in [\mathbb{N}]^{k+1}}$ be the Hamel basis of $c_{00}([\mathbb{N}]^{k+1})$. For every $x = \sum_{s \in [\mathbb{N}]^{k+1}} x(s)e_s$ in $c_{00}([\mathbb{N}]^{k+1})$, we set

$$(34) \quad \|x\| = \sup \left(\sum_{i=1}^n \|\mathcal{P}_i(x)\|_1^2 \right)^{\frac{1}{2}}$$

where $\|\mathcal{P}(x)\|_1 = \sum_{s \in \mathcal{P}} |x(s)|$, for all $\mathcal{P} \subseteq [\mathbb{N}]^{k+1}$ and the supremum in (34) is taken over all finite sequences $(\mathcal{P}_i)_{i=1}^n$ of disjoint Schreier plegmatic families in $[\mathbb{N}]^{k+1}$. The space \mathfrak{X}_{k+1} is defined to be the completion of $(c_{00}([\mathbb{N}]^{k+1}), \|\cdot\|)$.

The proof of the next proposition is straightforward.

Proposition 78. *The Hamel basis $(e_s)_{s \in [\mathbb{N}]^{k+1}}$ of $c_{00}([\mathbb{N}]^{k+1})$ is an unconditional basis for the space \mathfrak{X}_{k+1} and it generates a $(k+1)$ -spreading model which is isometric to the usual basis of ℓ^1 .*

We may also define a norming set W for the space \mathfrak{X}_{k+1} as follows. First, let

$$W^0 = \left\{ \sum_{s \in \mathcal{P}} \pm e_s^* : \mathcal{P} \subseteq [\mathbb{N}]^{k+1} \text{ is Schreier plegmatic} \right\}$$

For each $f = \sum_{s \in \mathcal{P}} e_s^* \in W^0$, the support of f , denoted by $\text{supp}(f)$, is defined to be the family \mathcal{P} . It is easy to see that a norming set for \mathfrak{X}_{k+1} is the set W which consists of all $f = \sum_{i=1}^n \lambda_i f_i$ where $(f_i)_{i=1}^n$ is a sequence in W^0 such that $\text{supp}(f_i) \cap \text{supp}(f_j) = \emptyset$, for all $1 \leq i < j \leq n$ and $\sum_{i=1}^n \lambda_i^2 \leq 1$.

In order to study the basic properties of the space \mathfrak{X}_{k+1} , we need the following proposition.

Proposition 79. *Every plegma disjointly generated k -spreading model of \mathfrak{X}_{k+1} is not equivalent to the usual basis of ℓ^1 .*

The proof is postponed in the next subsection. Assuming Proposition 79 we are able to prove the following.

Theorem 80. *The space \mathfrak{X}_{k+1} has the next properties.*

- (i) *It is reflexive.*
- (ii) *There is no sequence $(e_n)_n \in \mathcal{SM}_k(\mathfrak{X}_{k+1})$ equivalent to the usual basis of ℓ^1 .*
- (iii) *Every $(k+1)$ -subsequence of $(e_s)_{s \in [\mathbb{N}]^{k+1}}$ is not $(k+1)$ -Cesàro summable to any x_0 in \mathfrak{X}_{k+1} .*

Proof. (i) By Proposition 78, we have that $(e_s)_{s \in [\mathbb{N}]^{k+1}}$ is unconditional. Also, it is easy to check that it is boundedly complete. Thus c_0 is not contained in \mathfrak{X}_{k+1} . Moreover, the same holds for ℓ^1 , since otherwise there would exist a disjointly supported sequence $(x_n)_n \in \mathfrak{X}_{k+1}$ equivalent to the usual basis of ℓ^1 , which is impossible by Proposition 79. Hence, by James' theorem [12], the space \mathfrak{X}_{k+1} is reflexive.

(ii) Assume on the contrary, that there exists $(e_n)_n$ in $\mathcal{SM}_k(\mathfrak{X}_{k+1})$ equivalent to the usual basis of ℓ^1 . Since \mathfrak{X}_{k+1} is reflexive, we get that $(e_n)_n \in \mathcal{SM}_k^{wrc}(\mathfrak{X}_{k+1})$. Hence, by Corollary 34, $(e_n)_n$ is generated by a k -sequence $(x_s)_{s \in [\mathbb{N}]^k}$ in X_{k+1} admitting a canonical tree decomposition $(y_t)_{t \in [\mathbb{N}]^{\leq k}}$. Setting $x'_s = x_s - y_\emptyset$, for all $s \in [\mathbb{N}]^k$, by Lemma 37, we have that $(x'_s)_{s \in [\mathbb{N}]^k}$ also admits a k -spreading model equivalent to the usual basis of ℓ^1 . Since $(x'_s)_{s \in [\mathbb{N}]^k}$ is a plegma disjointly supported k -sequence, by Proposition 79 we have reached to a contradiction.

(iii) Since \mathfrak{X}_{k+1} is reflexive we have that $(e_s)_{s \in [\mathbb{N}]^{k+1}}$ is a weakly null $(k+1)$ -sequence.

Let $M \in [\mathbb{N}]^\infty$ and assume that $(e_s)_{s \in [M]^{k+1}}$ is $(k+1)$ -Cesàro summable to some $x_0 \in \mathfrak{X}_{k+1}$. By Proposition 56(ii), we get that $x_0 = 0$. For every $n \in \mathbb{N}$, let

$$(35) \quad y_n = \binom{(k+2)n}{k+1}^{-1} \sum_{s \in [M|(k+2)n]^{k+1}} e_s$$

where $l_n = (k+2)n$, $\mathcal{P}_n = F_1^n \times \dots \times F_{k+1}^n$, where for every $1 \leq i \leq k+1$, $F_i^n = \{M(in+1), \dots, M((i+1)n)\}$ and $f_n = \sum_{s \in \mathcal{P}_n} e_s^*$. It is easy to check that

$$(36) \quad f_n(y_n) = n^{k+1} \cdot \binom{(k+2)n}{k+1}^{-1} \xrightarrow{n \rightarrow \infty} \frac{(k+1)!}{(k+2)^{k+1}}$$

Since $\|y_n\| \geq f_n(y_n)$, by (36) we conclude that $(e_s)_{s \in [M]^{k+1}}$ is not $(k+1)$ -Cesàro summable to $x_0 = 0$, a contradiction. \square

12.2. Proof of Proposition 79.

Lemma 81. *Let $x \in \mathfrak{X}_{k+1}$ of finite support and $f \in W^0$ such that $\text{supp}(f) \cap \text{supp}(x) \neq \emptyset$. Then $|\text{supp}(f)| \leq n_0^{k+1}$, where $n_0 = \max\{s(1) : s \in \text{supp}(x)\}$.*

Proof. There exist $F_1 < \dots < F_{k+1}$ subsets of \mathbb{N} such that $|F_1| = \dots = |F_{k+1}|$, $\text{supp}(f) \subseteq F_1 \times \dots \times F_{k+1}$ and $|F_1| \leq \min F_1$. Hence $|\text{supp}(f)| \leq (\min F_1)^{k+1}$. Let $s \in \text{supp}(f) \cap \text{supp}(x)$. Then $n_0 \geq s(1) \geq \min F_1$. Hence $n_0 \geq \min F_1$ and therefore $|\text{supp}(f)| \leq n_0^{k+1}$. \square

Lemma 82. *Let $N_0 \in \mathbb{N}$. Then for every $0 < \varepsilon < 1$, every $l \in \mathbb{N}$ and every disjointly supported finite sequence $(x_j)_{j=1}^l$ in the unit ball of \mathfrak{X}_{k+1} such that for every $1 \leq j \leq l$ and $s \in \text{supp}(x_j)$, $s(1) \leq N_0$, we have that*

$$\left\| \frac{1}{l} \sum_{j=1}^l x_j \right\| \leq \varepsilon + \frac{N_0^{k+1}}{\varepsilon^2 l}$$

Proof. We fix $0 < \varepsilon < 1$, $l \in \mathbb{N}$ and $(x_j)_{j=1}^l$ satisfying the assumptions of the lemma. Let $\varphi = \sum_{i=1}^n \lambda_i f_i \in W$, where $n \in \mathbb{N}$, $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ with $\sum_{i=1}^n \lambda_i^2 \leq 1$ and $f_1, \dots, f_n \in W^0$ pairwise disjointly supported. For every $j = 1, \dots, l$ we set

$$I_j = \left\{ i \in \{1, \dots, n\} : \text{supp}(f_i) \cap \text{supp}(x_j) \neq \emptyset \right\}$$

By Lemma 81, we have that for every $1 \leq j \leq l$, if $i \in I_j$ then $|\text{supp}(f_i)| \leq N_0^{k+1}$. Also let $F_1 = \{j \in \{1, \dots, l\} : \sum_{i \in I_j} \lambda_i^2 < \varepsilon^2\}$ and $F_2 = \{1, \dots, l\} \setminus F_1$. It is easy to see that $\sum_{i \in I_j} \frac{f_i(x_j)}{(\sum_{i \in I_j} f_i(x_j)^2)^{\frac{1}{2}}} f_i$ belongs to W , for all $1 \leq j \leq l$. Hence, since $\|x_j\| \leq 1$, we have that $\sum_{i \in I_j} f_i(x_j)^2 \leq 1$, for all $1 \leq j \leq l$. Therefore we have

$$\begin{aligned} \varphi \left(\sum_{j=1}^l x_j \right) &= \sum_{i=1}^n \lambda_i f_i \left(\sum_{j=1}^l x_j \right) = \sum_{j=1}^l \sum_{i=1}^n \lambda_i f_i(x_j) \\ &= \sum_{j=1}^l \sum_{i \in I_j} \lambda_i f_i(x_j) \leq \sum_{j=1}^l \left(\sum_{i \in I_j} \lambda_i^2 \right)^{\frac{1}{2}} \left(\sum_{i \in I_j} f_i(x_j)^2 \right)^{\frac{1}{2}} \\ &\leq \sum_{j \in F_1} \left(\sum_{i \in I_j} \lambda_i^2 \right)^{\frac{1}{2}} + \sum_{j \in F_2} \left(\sum_{i \in I_j} \lambda_i^2 \right)^{\frac{1}{2}} \\ &\leq \varepsilon |F_1| + |F_2| \leq \varepsilon l + |F_2| \end{aligned}$$

If for some $1 \leq i \leq n$ we have that $J_i \neq \emptyset$ then, by Lemma 81 we have that $|\text{supp}(f_i)| \leq N_0^{k+1}$ and since $(x_j)_{j=1}^l$ are disjointly supported, we conclude that $|J_i| \leq N_0^{k+1}$. Therefore, for every $1 \leq i \leq n$, $|J_i| \leq N_0^{k+1}$. Hence

$$\varepsilon^2 |F_2| \leq \sum_{j \in F_2} \sum_{i \in I_j} \lambda_i^2 \leq \sum_{j=1}^l \sum_{i \in I_j} \lambda_i^2 = \sum_{i=1}^n |J_i| \lambda_i^2 \leq N_0^{k+1} \sum_{i=1}^n \lambda_i^2 \leq N_0^{k+1}$$

which yields that $|F_2| \leq N_0^{k+1}/\varepsilon^2$. Therefore, for every $\varphi \in W$ we have

$$\varphi\left(\sum_{j=1}^l x_j\right) \leq \varepsilon l + \frac{N_0^{k+1}}{\varepsilon^2}$$

Since W is a norming set for \mathfrak{X}_{k+1} , the proof is complete. \square

Definition 83. (i) Let $\mathcal{G}_1, \mathcal{G}_2 \subseteq [\mathbb{N}]^{k+1}$. We will call the pair $(\mathcal{G}_1, \mathcal{G}_2)$ weakly plegmatic if for every $s_2 \in \mathcal{G}_2$ there exists $s_1 \in \mathcal{G}_1$ such that the pair $\{s_1, s_2\}$ is plegmatic.

(ii) For every $0 \leq j \leq l$, let $\mathcal{G}_j \subseteq [\mathbb{N}]^{k+1}$. The finite sequence $(\mathcal{G}_j)_{j=0}^l$ will be called a weakly plegmatic path of subsets of $[\mathbb{N}]^{k+1}$, if for every $0 \leq i < l$ the pair $(\mathcal{G}_i, \mathcal{G}_{i+1})$ is weakly plegmatic.

Lemma 84. Let $(\mathcal{G}_j)_{j=0}^k$ be a weakly plegmatic path of subsets in $[\mathbb{N}]^{k+1}$. Then $\max\{s(1) : s \in \cup_{j=0}^k \mathcal{G}_j\} \leq \max\{s(k+1) : s \in \mathcal{G}_0\}$.

Proof. Let $0 \leq j \leq k$ and $s \in \mathcal{G}_j$. Then it is easy to see that there exists a sequence $(s_i)_{i=0}^j$ in $[\mathbb{N}]^{k+1}$ with $s_i \in \mathcal{G}_i$, for every $0 \leq i \leq j-1$ and $s_j = s$, such that $\{s_i, s_{i+1}\}$ is plegmatic, for all $0 \leq i \leq j-1$. Hence

$$s(1) = s_j(1) < s_{j-1}(2) < \dots < s_0(j+1) \leq s_0(k+1) \leq \max\{s(k+1) : s \in \mathcal{G}_0\}$$

\square

Lemma 85. Let $0 < \eta < \frac{1}{8}$ and $x_1, x_2 \in \mathfrak{X}_{k+1}$ with disjoint finite supports such that $\|x_1\|, \|x_2\| \leq 1$ and $\|x_1 + x_2\| > 2 - 2\eta$. Let $\mathcal{G}_1 \subseteq \text{supp}(x_1)$ such that $\|\mathcal{G}_1^c(x_1)\| \leq \eta$. Then there exists $\mathcal{G}_2 \subseteq \text{supp}(x_2)$ satisfying the following.

- (i) The pair $(\mathcal{G}_1, \mathcal{G}_2)$ is a weakly plegmatic path and
- (ii) $\|\mathcal{G}_2^c(x_2)\| \leq \eta^{\frac{1}{8}}$.

Proof. Since $\|x_1 + x_2\| > 2 - 2\eta$, there exists $\varphi \in W$ such that $\varphi(x_1 + x_2) > 2 - 2\eta$. Since $\|x_1\|, \|x_2\| \leq 1$, we get that $\varphi(x_1) > 1 - 2\eta$ and $\varphi(x_2) > 1 - 2\eta$. The functional φ is of the form $\sum_{i=1}^n \lambda_i f_i$, where f_1, \dots, f_n are pairwise disjoint supported elements of W^0 and $\sum_{i=1}^n \lambda_i^2 \leq 1$. We set $I = \{1, \dots, n\}$ and we split it to I_1 and I_2 as follows:

$$I_1 = \{i \in I : \text{supp}(f_i) \cap \mathcal{G}_1 \neq \emptyset\} \text{ and } I_2 = I \setminus I_1 = \{i \in I : \text{supp}(f_i) \subseteq \mathcal{G}_1^c\}$$

We also set $\varphi_1 = \sum_{i \in I_1} \lambda_i f_i$ and $\varphi_2 = \sum_{i \in I_2} \lambda_i f_i$. Hence $\varphi_2(x_1) \leq \|\mathcal{G}_1^c(x_1)\| \leq \eta$ and therefore $\varphi_1(x_1) > 1 - 3\eta$. Applying Cauchy-Schwartz's inequality we get that

$$1 - 3\eta < \varphi_1(x_1) = \sum_{i \in I_1} \lambda_i f_i(x_1) \leq \left(\sum_{i \in I_1} \lambda_i^2 \right)^{\frac{1}{2}} \left(\sum_{i \in I_1} f_i(x_1)^2 \right)^{\frac{1}{2}} \leq \left(\sum_{i \in I_1} \lambda_i^2 \right)^{\frac{1}{2}}$$

Since $\sum_{i \in I} \lambda_i^2 \leq 1$, we have that $(\sum_{i \in I_2} \lambda_i^2)^{\frac{1}{2}} < (1 - (1 - 3\eta)^2)^{\frac{1}{2}} \leq (6\eta)^{\frac{1}{2}}$. Hence

$$\varphi_2(x_2) = \sum_{i \in I_2} \lambda_i f_i(x_2) \leq \left(\sum_{i \in I_2} \lambda_i^2 \right)^{\frac{1}{2}} \left(\sum_{i \in I_2} f_i(x_2)^2 \right)^{\frac{1}{2}} < (6\eta)^{\frac{1}{2}}$$

Hence $\varphi_1(x_2) > 1 - 2\eta - (6\eta)^{\frac{1}{2}} > 1 - 4\eta^{\frac{1}{2}}$. We set $\mathcal{G}_2 = \text{supp}(x_2) \cap \text{supp}(\varphi_1)$. Then by the definition of I_1 it is immediate that the pair $(\mathcal{G}_1, \mathcal{G}_2)$ is weakly plegmatic. Finally, since $\|\mathcal{G}_2(x_2)\|^2 + \|\mathcal{G}_2^c(x_2)\|^2 \leq \|x_2\|^2 \leq 1$ and $\|\mathcal{G}_2(x_2)\| \geq \varphi_1(x_2)$, we get that $\|\mathcal{G}_2^c(x_2)\| \leq (1 - (1 - 4\eta^{\frac{1}{2}})^2)^{\frac{1}{2}} < \eta^{\frac{1}{8}}$ and the proof is complete. \square

An iterated use of the above yields the following.

Corollary 86. *Let $m \in \mathbb{N}$ and $0 < \varepsilon < \frac{1}{8}$. Then for every sequence $(x_i)_{i=0}^m$ of disjointly and finitely supported vectors in \mathfrak{X}_{k+1} with $\|x_i\| \leq 1$, for all $0 \leq i \leq m$, and $\|x_i + x_{i+1}\| > 2 - 2\varepsilon^{8^m}$, for all $0 \leq i < m$, there exists a weakly plegmatic path $(\mathcal{G}_i)_{i=0}^m$ of subsets of $[\mathbb{N}]^{k+1}$ such that $\mathcal{G}_i \subseteq \text{supp } x_i$ and $\|\mathcal{G}_i^c(x_i)\| < \varepsilon$, for all $0 \leq i \leq m$.*

We are now ready to give the proof of Proposition 79.

Proof of Proposition 79: Assume on the contrary that the space \mathfrak{X}_{k+1} admits a plegma disjointly generated k -spreading model equivalent to the usual basis of ℓ^1 . Let $0 < \varepsilon < \frac{1}{8}$. By Proposition 50 and Remark 11 there exists a sequence $(x_t)_{t \in [\mathbb{N}]^k}$ in the unit ball of \mathfrak{X}_{k+1} which is plegma disjointly supported and generates ℓ^1 as a k -spreading model of constant $c > 1 - \varepsilon^{8^k}$. Therefore, we may suppose that

$$(37) \quad \left\| \frac{1}{l} \sum_{j=1}^l x_{t_j} \right\| > 1 - \varepsilon^{8^k}$$

for all $l \in \mathbb{N}$ and $(t_j)_{j=1}^l \in \text{Plm}_l([\mathbb{N}]^k)$ with $t_1(1) \geq l$.

We set $t_0 = \{2, 4, \dots, 2k\}$, $N_0 = \max\{s(k+1) : s \in \text{supp}(x_{t_0})\}$ and $L = \{2n : s > k\}$. For every $t \in [L]^k$ we select $\mathcal{G}_t \subseteq [\mathbb{N}]^k$ such that $\mathcal{G}_t \subseteq \text{supp}(x_t)$, $\|\mathcal{G}_t^c(x_t)\| < \varepsilon$ and $s(1) < N_0$, for all $s \in \mathcal{G}_t$, as follows. Let $t \in [L]^k$. Observe $t \in [\mathbb{N}]_n^k$ and $t_0 < t$. By Proposition 9 there exists a plegma path $(t_j)_{j=0}^k$ in $[\mathbb{N}]^k$, with $t_k = t$. By Corollary 86 (for $m = k$) there exists a weakly plegmatic path $(\mathcal{G}_j)_{j=0}^k$ such that $\mathcal{G}_j \subseteq \text{supp } x_{t_j}$ and $\|\mathcal{G}_j^c(x_{t_j})\| < \varepsilon$, for all $j = 0, \dots, k$. We set $\mathcal{G}_t = \mathcal{G}_k$. Lemma 84 and Corollary 86 yield that the choice of $(\mathcal{G}_t)_{t \in [L]^k}$ is as desired.

For every $t \in [L]^k$, let $x_t^1 = \mathcal{G}_t(x_t)$. Then $\|x_t - x_t^1\| < \varepsilon$, for all $t \in [L]^k$. Hence by (37) we get that for every $l \in \mathbb{N}$ and every $(t_j)_{j=1}^l \in \text{Plm}_l([L]^k)$ with $t_1(1) \geq l$, we have that

$$(38) \quad \left\| \frac{1}{l} \sum_{i=1}^l x_{t_i}^1 \right\| > 1 - 2\varepsilon > \frac{6}{8}$$

Moreover notice that $(x_t')_{t \in [L]^k}$ is a plegma disjointly supported k -subsequence in the unit ball of \mathfrak{X}_{k+1} . Therefore, by Lemma 84 and (38) for $l > 8N_0^{k+1}/5\varepsilon^2$, we get a contradiction. The proof of Proposition 79 is complete. \square

Remark 16. As we have mentioned in the introduction of this article, the k -spreading models of a Banach space X have a transfinite extension yielding an hierarchy of ξ -spreading models, for $\xi < \omega_1$. It can be shown that the space in Section 11 does not admit ℓ^p , for $1 \leq p < \infty$, or c_0 as ξ -spreading model, for

every $\xi < \omega_1$. Also an analogue of the last example exists. Namely, for every limit countable ordinal ξ there exists a reflexive space X_ξ admitting ℓ^1 as ξ -spreading model but not less.

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